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DISPERSION RELATIONS AND THE ABSTRACT APPROACH TO FIELD THEORY

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DISPERSION RELATIONS AND THE ABSTRACT APPROACH TO FIELD THEORY

Edited by
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National Bureau of Standards

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PREFACE

It often happens in physics that a field grows with such rapidity that it becomes impossible to collect and organize the developments into a book before they are superseded by newer results. The present volume is an attempt to fill the need for an introduction to such a field: that of the axiomatic approach to field theory and its application to proving dispersion relations. The basic papers as well as the more recent advances have been included here in an attempt to present a coherent picture of the development of a field which at first sight seems chaotic. It is hoped that the reader will be able to find here the fundamental material to enable him to acquire a sufficiently broad knowledge of this new approach to elementary particle physics to read the new material which is now appearing in almost every issue of the journals.

It is, moreover, often easier to understand a new theory if one reads the original papers written when there was considerable confusion as to which path to follow in arriving at the correct formulation. After the theory reaches a more advanced stage of development, the literature tends to become less detailed in the treatment of the foundations and hence difficult to follow in the development of the physical content even though the logic becomes more powerful. It is hoped, therefore, that this volume—apart from any historical value it may have—will serve to illustrate the difficulties which exist during the formation of a new physical theory, and that it will point up more sharply the problems and complexities involved in the growth of physics.

Field Theory in recent years has tended to follow two independent lines in the approach to Elementary Particle Physics.

One approach has been to treat the strong interactions as, though pathological, fundamentally akin to the weak interactions. Quantum Electrodynamics, with its perturbation expansions and renormalization procedure, has been the model for further developments. The hope of this approach is that new physical ideas, developing from the mass of data being delivered by the huge new accelerators with their bubble chambers and related apparatus, will enable one to modify the existing field theories, and thus ultimately lead to the successful description of elementary particles. However, although Quantum Electrodynamics is

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able to attain a fantastic degree of accuracy in describing experiments involving weak interactions, the strong interactions have thus far failed to yield to this type of analysis.

The other, newer approach to the interpretation of the elementary particles utilizes a few general principles drawn from the experience with previous field theories and attempts a systematic study of the mathematical implications of these assumptions. The view here is that there is perhaps already enough experimental data to construct a field theoretic description if one were clever enough. This is the Abstract Approach to Field Theory, or Asymptotic Mechanics as it is sometimes called. Here, one attempts to avoid all the divergencies which arise from splitting a Hamiltonian into an unperturbed part plus a perturbation. In fact, no Hamiltonian is introduced at all. In addition, all fields introduced are already renormalized, and all particles are "dressed." No infinities should arise in this theory. This is a pure S-matrix type of theory, where one does not consider the details of the interaction at all, but merely the consequences of certain restrictions on the transformation from initial to final states.

This volume, contains the original papers which developed this latter approach. In particular, the axiomatic approaches of Lehmann, Symanzik, and Zimmermann, and A. S. Wightman have been included here since they are considered to be the formulations which are most fruitful.

It is important to keep in mind, however, that as yet no one has been able to calculate any physical processes with any of these formalisms. Only general theorems about field theory have been proved and it is not yet known whether or not these sets of axiomatics are empty. The principal use of these approaches has been in the proofs of the dispersion relations, i.e., in providing spectral representations and establishing the analyticity properties of matrix elements of the S-matrix. In this way, statements can be made about the scattering amplitudes for certain processes. The articles included in this volume besides developing the axiomatic formalism provide an excellent introduction to its principle application, the proofs of the dispersion relations.

The first paper in this volume, by H. Lehmann, introduces the general methods of proof used in Abstract Field Theory. With the technique of spectral representations developed here, Lehmann is able to prove that the propagators of the interacting fields are at least as singular as those of the free fields. The only assumptions needed to prove this surprising theorem were, aside from relativistic invariance, the completeness of the set of positive energy states and the absence of negative energy states. This result meant that there is no hope of removing the singularities in the calculation of observable quantities by writing down the correct interacting fields, and hence, no hope of eliminating the difficulties in perturbation expansions.

The two papers which follow by Lehmann, Symanzik and Zimmermann comprise one of the first completely abstract formulations of field theory. Here one attempts to circumvent all the difficulties involved in the writing down of interacting fields by assuming a completely renormalized, dressed particle with a complete set of states which form the elements of a Hilbert space and is described by an ordinary field operator, $A\ (x)$. This operator must be Lorentz invariant and must satisfy two important conditions. First is the so-called micro-causality condition which requires that the commutator, $[A(x),\ A(y)]$ must vanish on spacelike surfaces, that is, when $(x—y)^2>0$. This means that two points cannot affect each other with an interaction which propagates faster than the speed of light no matter how close the points may be. This requirement insures that the resulting field theory will be causal since the field operators which determine the observables of the theory must themselves be causal.

The second condition introduces the physics into the mathematical framework. This is the asymptotic condition which, qualitatively, says that earlier in the infinite past and later in the infinite future, the particles involved in a scattering process do not interact. That is, at $t = -\infty$ and $t = +\infty$ the fields, $A\,(x)$, are free fields. The precise mathematical formulation of this condition is far from simple, and this problem has been the basis for extensive discussion in the literature.

These postulates form the foundation for a theory which represents the first real extension of the S-matrix theory proposed by Heisenberg in 1943. With these axioms it is possible to derive a reduction formula which enables one to write the S-matrix entirely in terms of vacuum expectation values of time-ordered products of the field operators (the τ functions). Furthermore, a relation is found with which one can write down one of these τ functions in terms of all the other τ functions of the theory. Thus, an infinite set of equations which determine completely the τ functions is found. If these equations could be solved, a test of the formalism could be found. Unfortunately, this set of equations remains unsolved at the present time and it is entirely an open question whether they are merely an identity or whether they contain physical restrictions on the theory.

The paper following, by A. S. Wightman, outlines the "Wightman

Program" in Abstract Field Theory, the most ambitious attempt in axiomatics to date. In an extremely precise mathematical manner the postulates which were outlined above are formulated. The vacuum expectation values of products of field operators (the Wightman functions) are expressed by the powerful apparatus of distribution theory so as to avoid the difficulties associated with singular functions. The field operators, themselves, are expressed by the classifications of representations of the Lorentz Group. The result proved in this paper is that a local relativistic field theory is completely characterized by the set of Wightman functions. The equation for the τ functions and the expression for the S-matrix found in the previous papers by Lehmann, Symanzik, and Zimmermann are clearly a special case of this more general result.

One of the most remarkable theorems in axiomatics is proved in the next paper, by R. Haag. He shows here that any field related at one time to a free field by a unitary transformation must be equivalent to a free field throughout the space-time continuum. This does not preclude the existence of an S-matrix theory, but forces one to be extremely subtle in the definitions in this type of theory. The paper by Hall and Wightman puts much of the work in the previous reference into a rigorous mathematical framework.

In all the preceding work the bound states of the particles have been neglected. The paper by Zimmermann discusses the modifications necessary for these considerations.

The next group of papers is concerned with the application of this abstract formulation to the problem of proving dispersion relations. The axiomatic method permits one to prove certain spectral representations of matrix elements of the S-matrix. These representations, called dispersion relations, describe, of course, the amplitudes of arbitrary transitions in scattering processes. The reduction formula given in the second Lehmann, Symanzik, and Zimmermann paper can be used to write essentially these matrix elements in terms of the commutator of two field operators. This method transcribes automatically all of the axioms from the abstract formulations directly into properties of the scattering amplitudes. The fact that these commutators vanish outside the light cone enables one to continue analytically the amplitudes into the complex plane. This extension is not simple and must be done in a very subtle manner. Bremmermann, Oehme, and Taylor have been able to develop proofs founded on the theory of many complex variables for dispersion relations obtained in this way.

R. Jost and H. Lehmann earlier found a spectral representation for the special case of equal mass scattering, which represented an important advance in the method of proof of dispersion relations. A similar representation was found for the more general case by F. J. Dyson, thus permitting the proofs of dispersion relations to be done by more familiar mathematical methods. The paper following by Lehmann makes use of the Dyson representation for proving analytic continuation and also gives a proof of the convergence of the Legendre polynomial expansion of the imaginary part of the amplitudes when the range of the argument is unphysical. This enables one to show that the imaginary parts of the amplitudes retain their meaning in the extension into the non-physical region. The three papers mentioned before along with this paper by Lehmann constitute an excellent introduction to all essential steps for the proofs of dispersion relations.

Beyond the problem of proving the dispersion relations lies the question of how to predict experimental phenomena from them. The method of extracting this information is based on the presence of singularities in the scattering amplitudes, since these poles govern the behavior of the amplitudes in their neighborhood. The search for the location and classification of these poles is known as polology.

Perturbation theory has been extremely useful in polology because poles that have not been derived on a more rigorous basis are suggested by Feynman diagrams of the scattering process. A simple pole, for example, will occur in a scattering amplitude when the conservation laws permit a single particle intermediate state. A branch point occurs at the point where production amplitudes compete with scattering amplitudes and real particles can occur in the intermediate states. The case of scattering with fixed momentum transfer has been exhaustively studied, but only recently has the more general case become amenable to any kind of analysis and this, due to the spectacular success of a conjecture by S. Mandelstam. The representation for arbitrary energy-momentum transfer which Mandelstam wrote down enables one to extend both complex variables simultaneously into the complex plane. The calculations based on the Mandelstam representation, done with the perturbation series approach, enable one to write a complete dynamical description of strong interaction scattering.

The paper by R. J. Eden discusses the problem of finding a proof of the Mandelstam representation. Eden has recently been investigating the proof of the conjecture within the framework of perturbation theory and encountered certain difficulties in the sixth-order diagram. In any case, the general proof of this representation is extremely difficult and remains to be given.

As an illustration of the scattering theorems that may be proved using dispersion relations, the paper by I. Pomeranchuk contains the proof of the remarkable theorem that in the high energy limit if the π^+ and π^- cross sections approach a constant, they will become equal.

This volume concludes with a rather long review article by S. Gasiorowicz in which the methods of proof and application of the dispersion relations is treated at length. This final article, rounding out the entire group, provides a summary and presents the outlook for the future of this new branch of field theory.

It is not to be expected, of course, that all of the important work in Abstract Field Theory should be included here. This is only a selection of those papers which seem to contain the most fruitful approaches or results, a selection inevitably reflecting the personal bias of the editor. The method of selection employed, however, and the emphasis throughout the volume, has been on those papers which would present a survey of the fundamental physical ideas and mathematical techniques adequate to permit the reader to become conversant with this new and rapidly changing field of physics.

I wish to express my appreciation to James Siagas and Paula Siagas for their translation of the papers from the German.

April, 1961 LEWIS KLEIN

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DISPERSION RELATIONS
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PROPERTIES OF PROPAGATION FUNCTIONS AND RENORMALIZATION CONSTANTS OF QUANTIZED FIELDS

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Summary. It is attempted to derive some general properties of the propagation functions for coupled fields (Δ'_F, S'_F) without the use of power series expansions, and to show their connection with the renormalization constants for field operators and masses. Assuming that the coupled functions exist, it appears possible to discuss their behavior near the light cone (or for large momenta) and to obtain some information about the singularities of these functions when continued analytically. Attempts at the treatment of renormalizable theories are criticized on the basis of these results. Formulas are given for the mentioned renormalization constants which contain inequalities for the constants Z_2 and Z_3. Finally, it is pointed out that the methods introduced are advantageous for computations by means of power series expansions. As an example, the lowest order correction to the S_F-function in pseudoscalar meson theory is calculated without the appearance of infinite terms during the calculation.

INTRODUCTION

In recent work which treats the interaction problem in the structure of quantum field theory, the values Δ'_F and S'_F designated as propagation functions (or as Greens functions) play a significant role. These values should in principle be calculated from the fundamental equations of the theory. As yet however, only pertubation theoretical approximations are known, which probably with the exception of Quantum Electrodynamics are completely insufficient. It appears therefore suitable to obtain statements about these functions without assuming the possibility of their application with a coupling parameter or so using them.

The main results are therefore the derivation of formulas, which make possible the representation of these functions as the superposition of propagation functions of free fields with different masses and conclusions resulting from above. These concern especially the behavior of $\Delta'_F(x)$ and $S'_F(x)$ functions for small values of x^2, for example, the behavior of their Fourier transform for large momentum.

In the second part it will be shown that the constants appearing by the renormalization of field operators and masses can be expressed in a simple way through the values introduced for representing the propagation functions. One obtains in this way equations that are independent of pertubation computations and to a few expressions about these constants.

In contrast to other representations, in this fashion the use of incoming fields is avoided. In conclusion for illustration of methods used we will take up an analysis of pertubation theory. Two things should be pointed out: One, so far, it is not known if the basic equations of some quantized field theories (with the exception of free fields) possess solutions. We do not pursue this question but we strive for expressions of the propagation functions under the assumption that these exist. The other point is that to derive general results it is indispensable to operate with functions that are not explicitly known. Some of the considered mathematical operations (especially interchange of order of integration) have therefore formal character; their correctness could be shown only through performance of detailed calculations.

I. PROPERTIES OF PROPAGATION FUNCTIONS

a) Scalar fields:

In order to investigate the properties of propagation functions it appears useful to introduce other functions besides Δ'_F all of which can be defined as Vacuum Expectation values of the Heisenberg operators. The situation here is very similar as in a free field, where in a customary way one can derive Δ as well as $\Delta^{(1)}$ functions and from them easily change into Δ_F function.

Next, a Hermitian scalar field $A(x)$ will be treated, that can be coupled with itself in a non-linear way or may be in a position of interacting with other (Bose or Fermion) fields. No special assumptions shall be made about the manner of coupling, it can be local or non-local. It is assumed that the treated theory is Lorentz invariant. As a result all energy-momentum four vectors P_μ should exist with the property:

$$\frac{\partial A(x)}{\partial x_\mu} = i\left[A(x), P_\mu\right]; \qquad \left[P_\mu, P_\nu\right] = 0. \tag{1}$$

Furthermore it is assumed that it is possible to define a vacuum state: that is the energy operator should possess a smallest eigenvalue which we normalize to zero. The knowledge of commutation relations for field operators is almost not necessary. Equation (1) is sufficient.

Now we want to examine the vacuum expectation values of quadratic field magnitudes. As the orthogonal system in Hilbert space we shall use in this case, the eigenvectors of the P_μ operators, which should form a complete set.

It follows then

$$P_\mu \Phi_k = k_\mu \Phi_k \qquad (k_0 \geqslant 0). \tag{2}$$

The eigenvalues k_μ can naturally be degenerate.

In complete analogy to free fields we have the following:

$$\left|\begin{aligned} (\Phi_0, A(x)A(x')\Phi_0) &= \langle A(x)A(x')\rangle_0 = i\Delta^{(+)'}(x-x') \\ \langle A(x')A(x)\rangle_0 &= -i\Delta^{(-)'}(x-x') \\ \langle [A(x), A(x')]\rangle_0 &= i\Delta'(x-x') = -2i\varepsilon(x_0-x_0')\bar{\Delta}'(x-x') \\ \langle \{A(x), A(x')\}\rangle_0 &= \Delta^{(1)'}(x-x') \\ \langle TA(x)A(x')\rangle_0 &= \tfrac{1}{2}\Delta_F'(x-x') . \end{aligned}\right. \tag{3}$$

From these defining equations of vacuum functions it follows that the same relations exist among them as in a free field. For example one can express all functions through $\Delta^{(+)'}$. For the propagation functions one can state: $[\theta(x_0) = \tfrac{1}{2}(1 + x_0/|x_0|)]'$

$$\Delta_F'(x) = 2i\left[\theta(x_0)\Delta^{(+)'}(x) - \theta(-x_0)\Delta^{(-)'}(x)\right] = \Delta^{(1)'}(x) - 2i\bar{\Delta}'(x) . \tag{4}$$

The functions $\Delta^{(+)'}$ or $\Delta^{(-)'}$ contain only positive or negative frequencies (see below) so that Δ_F' can be interpreted as a causal function in the same manner as in a free field. To obtain an explanation of the structure of these functions we consider the $\Delta^{(+)'}$ function.

$$\begin{aligned} \langle A(x)A(x')\rangle_0 &= \sum_k (\Phi_0, A(x)\Phi_k)(\Phi_k, A(x')\Phi_0) \\ &= \sum_k A_{0k}(x)A_{k0}(x) = \sum_k a_{0k}a_{0k}^* \exp[ik(x-x')]. \end{aligned} \tag{5}$$

Here we set $(\Phi_0, A(x)\Phi_k = A_{0k}(x) = a_{0k}\, e^{ikx}$

The possibility of this conversion follows in the known way from

$$\frac{\partial A_{0k}(x)}{\partial x_\mu} = i(\Phi_0, [A(x), P_\mu]\Phi_k) = ik_\mu A_{0k}(x).$$

The summation in (5) is to extend over all states.

We introduce now a function

$$\varrho(-k^2) = (2\pi)^3 \sum a_{0k}a_{0k}^* . \tag{6}$$

Here we sum over all states that belong to eigenvalue k_μ.

It follows now from equations (3), (5) and (6)[1]

$$\Delta^{(+)'}(x-x') = -\frac{i}{(2\pi)^3}\int \theta(k_0)\varrho(-k^2)\exp[ik(x-x')]\,d^4k . \tag{7}$$

By this the summation over the eigenvalues is replaced by an integration with the condition that for all eigenvalues, $k_0 \geq 0$ is true.

We place in (7)

$$\varrho(-k^2) = \int_0^\infty \varrho(\varkappa^2)\,\delta(k^2+\varkappa^2)\,d(\varkappa^2)\,,$$

and obtain

$$\Delta^{(+)\prime}(x) = \int_0^\infty \Delta^{(+)}(x;\varkappa^2)\varrho(\varkappa^2)\,d(\varkappa^2)\,. \tag{8}$$

Analogous formula apply for all vacuum functions (in the following denoted as $\Delta^{(\ )\prime}$) since they can be formed in a linear way from $\Delta^{(+)\prime}$ according to (3).

$$\Delta^{(\ )\prime}(x) = \int_0^\infty \Delta^{(\ )}(x;\varkappa^2)\varrho(\varkappa^2)\,d(\varkappa^2)\,. \tag{9}$$

The primed functions allow themselves to be represented through a mass density by means of a linear transformation of the corresponding free functions.[2]

We have in the derivation of formula (9) already used that ρ is only dependent on argument k_μ^2 and for $k_\mu^2 > 0$ is identically zero. Both properties follow from the Lorentz invariance. Otherwise the vacuum functions (especially the commutation function Δ') will not be invariant quantities. Furthermore it follows from the defining equation (6) that ρ is a positive function. It is true

$$\varrho(\varkappa^2) \geqslant 0\,. \tag{10}$$

For the special case of a free field of mass m, it is naturally $\rho(\kappa^2) = \delta(\kappa^2 - m^2)$.

In general the discrete eigenvalues of the operators P_μ^2 will give rise to δ functions in $\rho(\kappa^2)$ inasmuch as the matrix element of the operators A(x) between the vacuum and the corresponding state (i.e. on the basis of selection rules) do not disappear. The discrete eigenvalues of P_μ^2 comply with the stable particles that were developed from the theory. From a physically meaningful theory one would expect that it at least describes a stable particle, namely ρ will contain at least one δ function. If no other stable particles appear, then it is true

$$\varrho(\varkappa^2) = \delta(\varkappa^2 - m^2) + \sigma(\varkappa^2); \qquad \sigma(\varkappa^2) = 0 \quad \text{for} \quad 0 \leqslant \varkappa^2 < (2m)^2;$$

where $\sigma(\kappa^2)$ is free of δ functions.

This structure describes the circumstance that in one such case the operator P_μ^2 possesses a discrete eigenvalue, to which a continuum fixes itself, when at least two particles are present.[3] From equation (9) and from the just given property of function ρ we can draw several conclusions. First of all it is clear that (9) is true also in Fourier space. ρ is therefore up to a certain factor equal to the transform of $\Delta^{(1)\prime}$ functions.

$$\varrho(-k^2) = \frac{1}{2\pi}\,\Delta^{(1)\prime}(k^2)\,. \tag{11}$$

Furthermore it is true for example:

$$\Delta_F'(k^2) = -2i\int_0^\infty \frac{\varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2)}{k^2+\varkappa^2-i\varepsilon} \tag{12}$$

With the given relations we can now disregard the behavior of $\Delta^{(\;)\prime}$ functions in the neighborhood of the light cone, or for their Fourier transforms having large values of k^2 because the behavior of "free" functions is known and does not depend on mass. So one obtains for example:

$$\bar{\Delta}'(x) = \frac{1}{4\pi}\int_0^\infty \{\delta(x^2)+\dots\}\varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2) = \frac{1}{4\pi}\,\delta(x^2)\int_0^\infty \varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2)+\dots \tag{13}$$

and in momentum space

$$\bar{\Delta}'(k^2) = P\int_0^\infty \frac{\varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2)}{k^2+\varkappa^2} = \frac{1}{k^2}\int_0^\infty \varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2)+\dots \tag{14}$$

if the separated integrals converge.

The primed functions have either the same behavior as the corresponding free functions or in case $\int_0^\infty \rho(\kappa^2)\mathrm{d}(\kappa^2)$ is not convergent—they are more strongly singular at the light cone (if they fall off more weakly for large momenta) than the free functions.

But it is not possible that the primed functions are less singular than the free functions because, from (10), $\int_0^\infty \rho\,\mathrm{d}(\kappa^2) > 0$.

It will be shown that the question of convergence of $\int_0^\infty \rho\,\mathrm{d}(\kappa^2)$ for theories capable of being renormalized is the same as the question whether the renormalization constants for the field operators are finite.

That the function Δ_F' should exist at all, it is, according to (12), clearly necessary, that the integral

$$\int^\infty \frac{\varrho(\varkappa^2)}{\varkappa^2}\,\mathrm{d}(\varkappa^2)\,,$$

should converge at the upper limit. A further conclusion concerns the properties of the function $\Delta_F'(k^2)$ by analytical continuation in k_μ^2 plane.

It is for real k^2

$$\Delta_F'(k^2) = \Delta^{(1)\prime}(k^2) - i2\bar{\Delta}'(k^2).$$

Furthermore it is true

$$2\bar{\Delta}'(k^2) = \frac{1}{\pi} P \int_0^\infty \frac{\Delta^{(1)\prime}(l^2)\,\mathrm{d}(-l^2)}{k^2 - l^2}\,; \quad \Delta^{(1)\prime}(k^2) = -\frac{1}{\pi} P \int_{-\infty}^\infty \frac{2\bar{\Delta}'(l^2)\,\mathrm{d}(-l^2)}{k^2 - l^2}\,(^4)\,. \tag{15}$$

The functions $\Delta^{(1)\prime}(k^2)$ and $2\overline{\Delta}'(k^2)$ are conjugates in the sense of Hilbert transformation.[5] From this (under certain assumptions) the function $\Delta_F'(k^2)$ can be analytically continued and is regular in the lower half plane. This result is gratifying because of the behavior of the pole in this region could lead to new (nonrenormalizable) divergence (i.e. of S matrix elements).[6]

Our analysis shows that by using the exact Δ_F' function, such difficulties should not arise. This is true also for the similarly discussed S_F' functions.

We want now to draw a conclusion for the commutation relations at equal times and use for this purpose the presentation (9) for the Δ' function.

$$\Delta'(x) = \int_0^\infty \Delta(x; \varkappa^2)\,\varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2)\,.$$

It follows directly:

$$\left|\begin{aligned} &\langle [A(\boldsymbol{x}, t), A(\boldsymbol{x}', t)] \rangle_0 = \langle [\dot{A}(\boldsymbol{x}, t), \dot{A}(\boldsymbol{x}', t)] \rangle_0 = 0\,, \\ &\langle [\dot{A}(\boldsymbol{x}, t), A(\boldsymbol{x}', t)] \rangle_0 = -i\delta(\boldsymbol{x} - \boldsymbol{x}') \int_0^\infty \varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2)\,. \end{aligned}\right. \tag{16}$$

The commutation relations at equal times between operators A and $\dot{A}$ for the vacuum state can be therefore derived (up to a factor) from equation (1).

The operators commute for spacelike points, and the Dirac δ function necessarily occurs. This is true, according to their derivation also for theories with non-local interactions.[7] The same state of affairs will occur for spinor fields.

We remark for equation (16) that according to the usual assumption for the commutation relation $[\dot{A}(\mathbf{x}), A(\mathbf{x}')] = -i\delta(\mathbf{x} - \mathbf{x}')$, the following will be true:

$$\int_0^\infty \rho\,\mathrm{d}(\kappa^2) = 1.$$

The vacuum functions that were here considered should however relate to renormalized operators, to which, as is known, one must transform.

b) Spinor fields

We now consider a spinor field under the same assumptions as were made for the already treated scalar field. Besides this we require the invariance of the theory toward the particle-antiparticle conjugation. The procedure and the results are accordingly similar as in a scalar field.

Again we begin with the definition of vacuum functions:

$$\left\{\begin{array}{l} \langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_0 = -iS^{(+)'}_{\alpha\beta}(x-x'); \qquad \langle\bar{\psi}_\beta(x')\psi_\alpha(x)\rangle_0 = -iS^{(-)'}_{\alpha\beta}(x-x') \\ \langle\{\psi_\alpha(x),\bar{\psi}_\beta(x')\}\rangle_0 = -iS'_{\alpha\beta}(x-x'); \qquad \langle[\psi_\alpha(x),\bar{\psi}_\beta(x')]\rangle_0 = -S^{(1)'}_{\alpha\beta}(x-x') \\ \qquad\qquad \langle T\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_0 = -\tfrac{1}{2}\,S'_{F\alpha\beta}(x-x')\,. \end{array}\right. \tag{17}$$

From the invariance about particle-antiparticle conjugation it follows:

$$\langle\bar{\psi}_\beta(x')\psi_\alpha(x)\rangle_0 = \langle\bar{\psi}'_\beta(x')\psi'_\alpha(x)\rangle_0 = -C^{-1}_{\beta\gamma}\langle\psi_\gamma(x')\bar{\psi}_\delta(x)\rangle_0 C_{\delta\alpha}\,.$$

Therefore:

$$S^{(-)'}_{\alpha\beta}(x-x') = -C^{-1}_{\beta\gamma}S^{(+)'}_{\gamma\delta}(x'-x)C_{\delta\alpha}\,. \tag{17a}$$

Thus one can express all functions with $S^{(+)'}$. Now there is:

$$\langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_0 = \sum_k(\Phi_0,\psi_\alpha(x)\Phi_k)(\Phi_k,\bar{\psi}_\beta(x')\Phi_0) = \sum_k c^\alpha_{0k}\bar{c}^\beta_{0k}\exp[ik(x-x')]. \tag{18}$$

Next we should again perform the summation over those states that belong to eigenvalue k_μ. Accordingly we introduce two functions ρ_1 and ρ_2.

$$(i\gamma_{\alpha\beta}k - \sqrt{-k^2}\,\delta_{\alpha\beta})\varrho_1(-k^2) + \delta_{\alpha\beta}\varrho_2(-k^2) = -(2\pi)^3\sum c^\alpha_{0k}\bar{c}^\beta_{0k}\,. \tag{19}$$

Because of the relativistic invariance of the theory the expression (19) can depend on the γ-matrices only in the specified way. The division of the γ-free portion into ρ_1 and ρ_2 is of course arbitrary.

Exactly as in the scalar case it follows now

$$\begin{aligned} S^{(+)'}(x) &= -\frac{i}{(2\pi)^3}\int\theta(k_0)\{(i\gamma k-\sqrt{-k^2})\varrho_1(-k^2)+\varrho_2(-k^2)\}\exp[ikx]\,d^4k = \\ &= \int_0^\infty\{S^{(+)}(x;\varkappa)\varrho_1(\varkappa^2)+\Delta^{(+)}(x;\varkappa)\varrho_2(\varkappa^2)\}\,d(\varkappa^2)\,. \end{aligned} \tag{20}$$

A corresponding representation is true according to (17) and (17a) for all primed functions:

$$S^{(\;)'}(x) = \int_0^\infty\{S^{(\;)}(x;\varkappa)\varrho_1(\varkappa^2)+\Delta^{(\;)}(x;\varkappa^2)\varrho_2(\varkappa^2)\}\,d(\varkappa^2)\,. \tag{21}$$

For ρ_1 and ρ_2 it follows directly from (19) that both functions are real. Furthermore we want to prove the following inequalities.

$$\varrho_1(\varkappa^2)\geqslant 0; \qquad 0\leqslant\varrho_2(\varkappa^2)\leqslant 2\varkappa\varrho_1(\varkappa^2)\,. \tag{22}$$

We multiply in (19) on the left with $(i\gamma k+\alpha)$, on the right with $(i\gamma k+\alpha)\gamma_4$ and we obtain ($f_{0k}=(i\gamma k+\alpha)c_{0k}$)

$$\sum_\beta f^\beta_{0k}f^{*\beta}_{0k} = k_0\left[(\varkappa-\alpha)^2\varrho_1+2\alpha\varrho_2\right]\geqslant 0; \qquad (\varkappa^2=-k^2)\,.$$

This inequality is true for arbitrary real α. Because $k_0 > 0$ we have:

$$(\kappa - \alpha)^2 \rho_1 + 2\alpha\rho_2 \geqslant 0 . \tag{22a}$$

For $\alpha = 0$ it follows:

$$\rho_1 \geqslant 0 .$$

The minimum of equation (22a) as a function of α is obtained for $\alpha = (\kappa\rho_1 - \rho_2)/\rho_1$. Insertion of this value yields:

$$\rho_2(2\kappa\rho_1 - \rho_2) \geqslant 0 .$$

This leads directly to the specified inequalities for ρ_2. For a free spinor field we have: $\rho_1(\kappa^2) = \delta(\kappa^2 - M^2)$; $\rho_2(\kappa^2) \equiv 0$.

In general the function ρ_1 will contain at least one δ-function (since the theory should describe at least one stable particle of spin ½).

With the help of equations (21) and (22) one may now draw similar conclusions as in the scalar case. In the momentum-space we have for instance:

$$S'_F(k) = -2i \int_0^\infty \frac{(i\gamma k - \kappa)\rho_1(\kappa^2) + \rho_2(\kappa^2)}{k^2 + \kappa^2 - i\varepsilon} \, d(\kappa^2) . \tag{23}$$

Further we can again determine the behavior of the primed functions in the neighborhood of the light cone or for large momenta. Since $\rho_1 \geq 0$ the primed functions either have the same behavior as the free functions or if $\int_0^\infty \rho_1 d(\kappa^2)$ does not converge, they are more strongly singular.

The conclusions with regard to the behavior of the propagation function by means of analytical continuation can be drawn, similarly as in the scalar case. When one writes S'_F in the form $S'_F(k) = i\gamma k f_1(k^2) + f_2(k^2)$ then the real and imaginary part of functions $f_i(k^2)$ are conjugate to each other according to Hilbert-transformation. By continuation the functions will therefore be regular in the lower half-plane.

Finally one obtains from equation (21) for the anticommutator at equal times for the S'-function:

$$\left\langle \{\psi_\alpha(\boldsymbol{x}, t), \bar{\psi}_\beta(\boldsymbol{x}', t)\} \right\rangle_0 = \gamma^4_{\alpha\beta} \delta(\boldsymbol{x} - \boldsymbol{x}') \int_0^\infty \rho_1(\kappa^2) \, d(\kappa^2) . \tag{24}$$

c) Remark for Handling Non-Renormalizable Theories

We add here a remark with regard to the attempt[8] to obtain convergent results in nonrenormalizable theories. By this method one proceeds in such a way that for the calculation of S-matrix elements one substitutes for the Δ_F or S_F function, by means of approximation, certain Δ'_F or S'_F functions.

This process is based on the hope that on account of the field reaction the primed functions are less singular than the corresponding free functions. Actually one obtains with the usual calculation scheme (consideration of the irreducible self energy graph up to a certain order in the coupling constants) functions that fall off strongly for large momenta. Our results show however, that this is not the case for exact propagation functions; on the contrary the usual approximation method is inappropriate for representing the asymptotic behavior of the exact function. Therefore we do not believe that we can build up a consistent method from these attempts.

II. RENORMALIZATION CONSTANTS

The following section should serve the purpose to express the constants that appeared in the process of renormalization of the field operators and masses, through the previously introduced functions $\rho(\kappa^2)$. As an example of these relations we shall use the pseudoscalar coupling of a neutral meson-field A(x) to a spinor field.

From the usual Lagrange function the unrenormalized field operators ($A_u(x)$, $\psi_u(x)$), masses (M_0,m_0) and an unrenormalized coupling constant (g_0) included, occur through the introduction of a renormalized magnitude[9]

$$\left\{\begin{array}{lll} \psi(x) = Z_2^{-1/2}\cdot\psi_u(x); & A(x) = Z_3^{-1/2}A_u(x) & \\ M = M_0 + \delta M; & m^2 = m_0^2 + \delta m^2; & g = Z_1^{-1}Z_2 Z_3^{1/2} g_0\,, \end{array}\right. \tag{25}$$

$$L = \left|\begin{array}{l} -\tfrac{1}{4}Z_2\{[\bar{\psi},(\gamma\partial + M)\psi] - [(\gamma^T\partial - M)\bar{\psi},\psi]\} - \\ \qquad\qquad\qquad - \tfrac{1}{2}Z_3\{\partial_\nu A\partial_\nu A + m^2A^2\} \\ - i/2\, gZ_1[\bar{\psi},\gamma_5\psi]A + \tfrac{1}{2}Z_2\,\delta M[\bar{\psi},\psi] + \tfrac{1}{2}Z_3\delta m^2A^2\,. \end{array}\right. \tag{26}$$

The renormalized operators satisfy according to the field equations:

$$\left|\begin{array}{l} (\Box - m^2)A(x) = \dfrac{ig}{2}Z_1Z_3^{-1}[\bar{\psi}.\,\gamma_5\psi] - \delta m^2 A \\ \left(\gamma\dfrac{\partial}{\partial x} + M\right)\psi(x) = -\,igZ_1Z_2^{-1}\gamma_5A\psi + \delta M\psi\,. \end{array}\right. \tag{27}$$

The commutation and anticommutation relations at equal times are:

$$\left\{\begin{array}{l} [A(x),\dot{A}(x')] = -\,i\delta(\boldsymbol{x}-\boldsymbol{x}')\cdot Z_3^{-1} \\ \{\psi_\alpha(x),\bar{\psi}_\beta(x')\} = \gamma^4_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{x}')\cdot Z_2^{-1}. \end{array}\right. \tag{28}$$

All remaining commutators (or anticommutators) vanish for equal times.

According to (25) and the earlier given defining equations of functions $\rho(\kappa^2)$ the constants Z_3 or Z_2 come in simply as multiplicative factors in the functions ρ (or ρ_1 and ρ_2). Next we have to occupy ourselves with the question how the constants Z_2 and Z_3 are determined. Moreover we observe that the theory—in order to be physically meaningful—must describe at least two stable particles: namely the operator P_μ^2 must have a discrete eigenvalue for the nucleon number 0 with mass m (experimental mass of the mesons) and a discrete eigenvalue for the nucleon number 1 with mass M (experimental mass of the nucleons). Naturally only the states with nucleon number 0 or 1 give a contribution to the functions ρ. These functions contain therefore terms of form:

$$\varrho(\varkappa^2) = c_3\delta(\varkappa^2 - m^2) + \ldots; \qquad \varrho_1 = c_2\delta(\varkappa^2 - M^2) + \ldots$$

The constants Z_3 and Z_2 will now be determined so that we normalize to unity c_3 and c_2.[10]

This corresponds to the method of renormalization in pertubation theory[11] or to the renormalization conditions of Källen.[1]

For constants Z_3 and Z_2 it is true due to (16), (24) and (28):

$$Z_3^{-1} = \int_0^\infty \varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2); \qquad Z_2^{-1} = \int_0^\infty \varrho_1(\varkappa^2)\,\mathrm{d}(\varkappa^2)\,. \tag{29}$$

Since the functions ρ and ρ_1 are always positive and contain terms with $\delta(\kappa^2 - m^2)$ or $\delta(\kappa^2 - M^2)$ it is true, due to (29), for the renormalization constants that:

$$0 \leqslant Z_3 < 1; \qquad 0 \leqslant Z_2 < 1 \quad {}^{12}\,. \tag{30}$$

Now we want to obtain formulas for the mass constants δm and δM. We consider next the meson mass. Due to (9) and (27) we have:

$$\begin{aligned}(\Box - m_0^2)\,\langle[A(x), A(x')]\rangle_0 &= \frac{ig}{2} Z_1 Z_3^{-1} \langle[[\overline{\psi}(x), \gamma_5\psi(x)], A(x')]\rangle_0 = \\ &= i\int_0^\infty (\varkappa^2 - m_0^2)\Delta(x - x'; \varkappa^2)\,\varrho(\varkappa^2)\,\mathrm{d}(\varkappa^2)\,.\end{aligned} \tag{31}$$

We differentiate with respect to t' and then let $t' = t$. We have:

$$\langle[[\overline{\psi}(x), \gamma_5\psi(x)], \dot{A}(x')]\rangle_0 = 0 \qquad \text{for } t' = t$$

since for equal times:

$$[\psi(x), \dot{A}(x')] = [\overline{\psi}(x), \dot{A}(x')] = 0\,.$$

One obtains thus:

$$m_0^2 \int_0^\infty \varrho\,\mathrm{d}(\varkappa^2) = \int_0^\infty \varkappa^2 \varrho\,\mathrm{d}(\varkappa^2)\,,$$

or

$$\delta m^2 = -\frac{\int_0^\infty (\varkappa^2 - m^2)\varrho\,\mathrm{d}(\varkappa^2)}{\int_0^\infty \varrho\,\mathrm{d}(\varkappa^2)} = -Z_3 \int_0^\infty (\varkappa^2 - m^2)\varrho\,\mathrm{d}(\varkappa^2) \leqslant 0\,. \tag{32}$$

A formula for the nucleon constant δM is obtained in an analogous way.

$$\begin{aligned}\left(\gamma\frac{\partial}{\partial x}+M_0\right)S'(x-x') &= \\ &= \int\limits_0^\infty\left[(M_0-\varkappa)\gamma\frac{\partial}{\partial x}+\varkappa(\varkappa-M_0)\right]\Delta(x-x';\varkappa^2)\varrho_1(\varkappa^2)\,\mathrm{d}(\varkappa^2)+ \\ &+\int\limits_0^\infty\left(\gamma\frac{\partial}{\partial x}+M_0\right)\Delta(x-x';\varkappa^2)\varrho_2(\varkappa^2)\,\mathrm{d}(\varkappa^2)= \\ &= gZ_1Z_2^{-1}\gamma_5\left\langle\{A(x)\psi x),\overline{\psi}(x')\}\right\rangle_0\,,\end{aligned} \tag{33}$$

for t′ = t is

$$\left\langle\{A(x)\psi(x),\overline{\psi}(x')\}\right\rangle_0\sim\left\langle A(x)\right\rangle_0=0\,.$$

(The vanishing of $<A(x)>_0$ follows therefore from:

$$(\Box-m_0^2)\left\langle A(x)\right\rangle_0=-m_0^2\left\langle A(x)\right\rangle_0=\frac{ig}{2}Z_1Z_3^{-1}Sp\,\{\gamma_5S^{(1)\prime}(0)\}=0\,.$$

According to (32) $m_0^2 \neq 0$; therefore $<A(x)>_0 = 0$).

Hence it is true:

$$M_0\int\limits_0^\infty\varrho_1\,\mathrm{d}(\varkappa^2)=\int\limits_0^\infty(\varkappa\varrho_1-\varrho_2)\,\mathrm{d}(\varkappa^2)$$

or

$$\delta M=\frac{\int_0^\infty[(M-\varkappa)\varrho_1+\varrho_2]\,\mathrm{d}(\varkappa^2)}{\int_0^\infty\varrho_1\,\mathrm{d}(\varkappa^2)}=Z_2\int\limits_0^\infty[(M-\varkappa)\varrho_1+\varrho_2]\,\mathrm{d}(\varkappa^2)\,. \tag{34}$$

The formulas (29), (32) and (34) express the constants Z_2, Z_3, δm^2, δM by the functions $\rho(\kappa^2)$. We will not discuss the constant Z_1 here.

It may be evident, that the derived relations are not limited to the special case of the coupled pseudoscalar mesons with nucleons but can be applied to other cases. For an application to quantum electrodynamics certain supplementary conditions are required—on account of the Lorentz conditions. The results obtained in this section are similar to the definitions of renormalization constants for quantum electrodynamics by Kállen. The two presentations differ methodically in that here the use of incoming fields and the associated problems of the adiabatic interpolation can be avoided; to the other presentation we give credit on the relationship between renormalization constants and the properties of the propagation functions.

III. DETERMINATION OF THE VACUUM FUNCTIONS IN NEIGHBORHOOD OF g^2

Although our real wish is to obtain expressions for propagation functions that are independent of pertubation computations, we want to indicate that the methods used here are advantageous also for the moderate pertubation calculation of these functions. We illustrate these methods by determining the functions ρ_1, ρ_2 and S'_F in the neighborhood of g^2 for the previously considered example of the pseudoscalar coupling of mesons and nucleons. In contrast to other presentations no divergent constituents appear in this scheme, whose elimination by the otherwise common method gives rise to relatively intricate transformations.[13]

We set thus:

$$\varrho_1(\varkappa^2) = \varrho_{10}(\varkappa^2) + g^2\varrho_{11}(\varkappa^2) + \ldots$$
$$\varrho_2(\varkappa^2) = \varrho_{20}(\varkappa^2) + g^2\varrho_{21}(\varkappa^2) + \ldots$$

and determine these values by expanding for example the $S^{(+)\prime}(x)$ function in powers of g^2.

$$S^{(+)\prime}(x-x') = i\,\langle\psi(x)\bar{\psi}(x')\rangle_0 = S^{(+)}(x-x';\,M) + 0(g^2)\,. \tag{35}$$

In the neighborhood of higher orders of zero naturally the unprimed $S^{(+)}$ function occurs and therefore we have:

$$\varrho_{10} = \delta(\varkappa^2 - M^2)\,; \qquad \varrho_{20} = 0\,; \qquad \delta M_0 = 0\,; \qquad Z_{20} = 1\,.$$

To find the g^2 term of equation (35) we form

$$\begin{aligned}&\left(\gamma\frac{\partial}{\partial x} + M\right)\left(\gamma^T\frac{\partial}{\partial x'} - M\right)S^{(+)\prime}(x-x') = i\,\langle\, f(x)\bar{f}(x')\,\rangle_0 = \\ &= ig^2\gamma_5\,\langle A(x)\psi(x)A(x')\bar{\psi}(x')\rangle_0 + \ldots = ig^2\Delta^{(+)}(x-x';\,m)\gamma_5 S^{(+)}(x-x';\,M)\gamma_5 + \ldots\,.\end{aligned} \tag{36}$$

In (36) $f(x) = -igZ_1Z_2^{-1}\gamma_5 A(x)\psi(x) + \delta M\psi(x)$.

By further transformation only terms $\sim g^2$ are written out. By going over into Fourier space we have: ($\kappa^2 = -k^2$)

$$\begin{aligned}&\theta(k_0)(i\gamma k + M)\left[(i\gamma k - \varkappa)\varrho_{11} + \varrho_{21}\right](i\gamma k + M) = \\ &= -\frac{1}{(2\pi)^3}\int \theta(k_0 - q_0)\theta(q_0)\gamma_5(i\gamma q - M)\gamma_5\,\delta[q^2 + M^2]\,\delta\left[(k-q)^2 + m^2\right]\mathrm{d}^4q\,.\end{aligned} \tag{37}$$

In order to eliminate the γ-matrices we take the spur of (37). Another relation is obtained by multiplying (37) by $i\gamma k$ and taking the spur. We obtain

$$\left|\begin{aligned}&\theta(k_0)\left[-\varkappa(\varkappa - M)^2\varrho_{11} + (\varkappa^2 + M^2)\varrho_{21}\right] = \\ &\qquad = \frac{M}{(2\pi)^3}\int \theta(k_0 - q_0)\theta(q_0)\,\delta[q^2 + M^2]\,\delta\left[(k-q)^2 + m^2\right]\mathrm{d}^4q \\ &\theta(k_0)\varkappa^2\left[(\varkappa - M)^2\varrho_{11} + 2M\varrho_{21}\right] = \\ &\qquad = -\frac{1}{(2\pi)^3}\int \theta(k_0 - q_0)\theta(q_0)(kq)\,\delta[q^2 + M^2]\,\delta\left[(k-q)^2 + m^2\right]\mathrm{d}^4q.\end{aligned}\right. \tag{38}$$

The integrals that occur here are elementary to carry out and are convergent. We find for ρ_{11} and ρ_{21}:

$$\left|\begin{aligned} \varrho_{11} &= \frac{1}{32\pi^2}\,\theta\,[\varkappa^2-(M+m)^2]\cdot \\ &\qquad\cdot\frac{\sqrt{(\varkappa^2-M^2-m^2)^2-4m^2M^2}}{\varkappa^4}\left\{1-\frac{m^2(\varkappa^2+M^2)}{(\varkappa^2-M^2)^2}\right\} \\ \varrho_{21} &= \frac{1}{32\pi^2}\,\theta[\varkappa^2-(M+m)^2]\cdot \\ &\qquad\cdot\frac{\sqrt{(\varkappa^2-M^2-m^2)^2-4m^2M^2}}{\varkappa^3}\left\{1-\frac{m^2}{(\varkappa+M)^2}\right\}. \end{aligned}\right. \tag{39}$$

For $S'_F = S_F + g^2S'_{F1} + \cdots$ we have therefore

$$S'_{F1}(k) = -2i\int_0^\infty \frac{[(i\gamma k-\varkappa)\varrho_{11}+\varrho_{21}]\,d(\varkappa^2)}{k^2+\varkappa^2-i\varepsilon}\,.$$

To give $S'_{F1}(k)$ explicitly we have to carry out one more integral over κ^2. The integration is elementary to carry out and is trivial in the special case m = 0. For this case we give the result:[14]

$$S'_{F1}(k) = \frac{i}{16\pi^2}\,\frac{i\gamma k}{k^2}\left\{1-\left(1+\frac{M^2}{k^2}\right)\ln\left(1+\frac{k^2}{M^2}-i\varepsilon\right)\right\}. \tag{40}$$

It is established, in the above analysis of the S'_F function, that no divergent terms occur during the calculation. Naturally, one can determine the renormalization constants that did not appear here from ρ_{11} and ρ_{12} with the help of formulas (29) and (34).

In conclusion I would like to thank Prof. W. Heisenberg for many suggestions and K. Symanzik and W. Zimmerman for numerous discussions.

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10. Consider that the function ρ contains other δ-functions than those which correspond to stable particles with nucleon number 0 or 1. For a physically useful interaction theory of π-mesons and nucleons, these kinds of particles should not occur because they do not appear experimentally. In case they do occur in the theory then we must (by fixing Z_2 and Z_3) characterize precisely that discrete eigenvalue that belongs to the experimental meson or nucleon.
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THE FORMULATION OF QUANTIZED FIELD THEORIES

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Summary. A new formulation of quantized field theories is proposed. Starting from some general requirements we derive a set of equations which determine the matrix-elements of field operators and the S-Matrix. These equations contain no renormalization constants, but only experimental masses and coupling parameters. The main advantage over the conventional formulation is thus the elimination of all divergent terms in the basic equations. This means that no renormalization problem arises. The formulation is here restricted to theories which do not involve stable bound states. For simplicity we derive the equations for spin 0 particles, however the extension to other cases (e.g. quantum electrodynamics) is obvious. The solutions of the equations are discussed in a power-series expansion. They are then identical with the renormalized expressions of the conventional formulation. However, the equations set up here are not restricted to the application of perturbation theory.

I. INTRODUCTION

In order to clarify the position of the problem considered in this work, let us refer to some unsatisfactory features of the usual formulation of field theory:

1) While, for the so called renormalizable theories within the framework of a coupling constant expansion, it is possible to obtain finite results for the matrix elements of the (renormalized) field operators and for the S-matrix, so far no one has been successful in formulating the basic equations of those theories in a convergent fashion. We mean by this that divergent renormalization constants appear not only in the field equations but also in the commutation relations of renormalized operators.

2) These finite results can be obtained only by way of intermediate divergent calculations which are taken care of in the process of renormalization. If the prescription necessary for the separation of these divergent terms can be unambiguously formulated, they still are closely restricted to the perturbation treatment.

These conditions are detrimental not only for the treatment of fundamental questions but also in the application of the theory. First of all, they make it difficult to answer questions about the existence of solutions of the basic equations (as in quantum electrodynamics). Secondly, the necessary separation of (divergent) renormalization terms stands in the way of a consistent application to other approximations than the coupling constant expansion.

In our attempt at a new formulation of quantized field theories, we try to avoid the above described difficulties. An intrinsic characteristic of this formulation is that we derive equations that, in contrast to the usual basic equations, are free of divergence. In principle, they suffice for determination of matrix elements of the field operators and the S-matrix. The physically meaningless renormalization constants are completely eliminated from the theory; this means that not only in the basic equations but also in all further relations only renormalized field operators, experimental masses and coupling parameters occur, so that there is no renormalization problem. Field equations and canonical commutation relations of operators are not used. This renunciation appears indispensable when one wants to work only with renormalized quantities.

It is emphasized that we have in mind a new formulation, without changing the physical principles of quantum field theory. This manifests itself in the fact that the perturbation theory solutions of our equations are identical with the corresponding renormalized expressions of the usual formulation. In the derivation of these equations it is not assumed that the solutions can be expanded in a coupling parameter. To this extent an eventual divergent perturbation series means no fundamental difficulty for the discussion of the question of whether exact solutions of the basic equations exist.

We limit ourselves, for the time being, to the formulation of theories in which no stable bound particles appear. This should suffice, for example, for quantum electrodynamics. In order to have as simple relations as possible, we consider equations for the case where only one kind of particle is present, namely, particles with spin 0 and mass $\underline{m}$ whose interaction is to be described. This development can be readily carried over to other cases (i.e., quantum electrodynamics).

For the derivation of the equations used by us, only a few very general assumptions are necessary. Accordingly, these equations contain many solutions; the invariant solutions correspond to local field theories. The individual possibilities of interaction are characterized by boundary conditions imposed on the solutions of the equations. These solutions are for the time being considered only within the framework of perturbation calculations. We do not go into the treatment of non-local fields and into questions that depend on non-renormalizable theories.

II. GENERAL INFORMATION

We begin with the statement of general principles. Our aim is to give a quantum theoretical formulation of local scalar fields without bound states, which differs from the usual formulations in that, in principle, it avoids the difficulties of the ultraviolet divergences, namely, the basic equations of the theory should already be free from ultraviolet divergences. It is clear that for this purpose we cannot use such familiar relations as the Hamiltonian function of canonical field variables, the canonical commutation relations of the Feynman-Schwinger variational principle, since all these contain renormalization constants. We have to look for more general properties of quantized fields that are not burdened with such divergence difficulties and are therefore appropriate for the foundation of our theory. To this belongs the existence of an invariant, unitary S-matrix whose matrix elements being observable quantities must, by necessity, be finite. It is within this property that the framework of general quantized fields should be first considered. For characterizing the local fields, one must go beyond these considerations and formulate the causality condition in a suitable way. Since, so far, the causality condition as a property of the S-matrix has not been successfully formulated, we decide to introduce field operators and thereby go beyond the conception of a pure S-matrix theory.

We imagine a field described by a linear, Hermitian operator $A(x)$ in Hilbert space. We should now limit further the properties of the "field operator" $A(x)$. (It corresponds exactly to the so-called "renormalized field operator" of the earlier formulations.) This is accomplished by imposing the following three requirements:

1. Invariance principle. The theory should be invariant under Lorentz transformations, translations, as well as reflections, in space and time.

2. Causality condition. The space-like commutators of field operators should vanish:

$$[A(x), A(y)] = 0, \tag{1}$$

when $x-y$ is a space-like vector.

3. Asymptotic condition. The field operator $A(x)$ should for $x_0 \rightarrow -\infty$ and for $x_0 \rightarrow +\infty$ go over into the interaction free operators $A_{in}(x)$ or $A_{out}(x)$, respectively, for particle mass $\underline{m}$.

These three conditions on the field operator $A(x)$ will form the only assumptions of our theory of quantized scalar and local fields without bound states. Due to their significance, they shall now be briefly explained.

The invariance principle is so self-evident that a further discussion is unnecessary. As far as its mathematical formulation, one can consult the appropriate literature.[1]

The form of the causality condition[2] shows quite clearly that field effects at two points in a space-like interval are independent. This should be equivalent to other common causality concepts.[3]

Concerning the obvious meaning of the asymptotic condition[4] we can say: It reveals that in every system of interacting bosons, if one only waits long enough, the particles tend away from each other with increasing time, so that for large enough time they behave as interaction-free bosons of a certain mass m. Therefore, the asymptotic condition limits the theory to such fields that do not allow bound states but that describe only scattering, creation, and destruction of bosons of mass m. Since in the next chapter we shall draw very far-reaching conclusions, a precise mathematical formulation is indispensable. We begin with the definition of the incoming and outgoing fields. $A_{in}(x)$ and $A_{out}(x)$ should naturally satisfy the common interaction-free field equations and commutation relations:

$$(\Box - m^2)A_{in}(x) = 0\,, \qquad [A_{in}(x), A_{in}(x')] = i\Delta(x-x') \;\;(^5). \tag{2}$$

Further, there should be states with the property

$$A^+_{in}(x)\Omega_{in} = 0\,, \qquad A^+_{out}(x)\Omega_{out} = 0 \;\;(^6) \tag{3}$$

Special attention is required for the precision of the limiting conditions $x_0 \to \pm\infty$. A readily suggestible form is

$$\lim_{x_0\to-\infty} (\Phi, A(x)\Psi) = \lim_{x_0\to-\infty} (\Phi, A_{in}(x)\Psi)$$

but this does not say too much since the matrix elements of the right side go to zero point by point for normalizable states. We introduce instead, the operators

$$A^f(t) = i\iint\limits_{x_0=t}\left\{ A(x)\,\frac{\partial f(x)}{\partial x_0} - f(x)\,\frac{\partial A(x)}{\partial x_0}\right\} d_3x\,, \tag{4}$$

and corresponding operators $A^f_{in}(t)$ and $A^f_{out}(t)$. Here f(x) is an arbitrary normalized positive frequency solution of the Klein-Gordon equation, that is:

$$(\Box - m^2)f(x) = 0\,, \qquad -i\iint\left\{ f\,\frac{df^*}{\partial x_0} - f^*\,\frac{\partial f}{\partial x_0}\right\} d_3x = 1\,. \;\;(^7)$$

For the asymptotic condition we now have

$$\begin{cases} \lim\limits_{\tau\to-\infty} (\Phi, A^f(\tau)\Psi) = (\Phi, A^f_{in}(t)\Psi) \\ \lim\limits_{\tau\to+\infty} (\Phi, A^f(\tau)\Psi) = (\Phi, A^f_{out}(t)\Psi) \end{cases} \tag{5}$$

where the right sides are independent of t and thus are constant.

For the remainder of this section we derive consequences from the above three conditions.

From Ω_{in} we can form, by the repeated application of the operator $A_{in}(x)$, a complete orthonormal system in Hilbert space and similarly, a second system by application of $A_{out}(x)$ on Ω_{out}. For this, one needs only a system $f_\alpha(x)$ of solutions of the Klein-Gordon equation which contain only positive frequencies and obey the orthonormal relation

$$-i\int\left\{f_\alpha \frac{\partial f_\beta^*}{\partial x_0} - f_\beta^* \frac{\partial f_\alpha}{\partial x_0}\right\} d_3x = \delta_{\alpha\beta},$$

as well as the completeness relation

$$\sum_{\alpha=1}^{\infty} f_\alpha(x) f_\alpha^*(x') = i\Delta^+(x-x')$$

The two orthonormal systems are then:

$$\left|\begin{array}{l} \Omega_{\text{in}} \\ \Phi^{\alpha}_{\text{in}} = A^{\alpha}_{\text{in}}\Omega_{\text{in}} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \Phi^{\alpha_1\ldots\alpha_k}_{\text{in}} = \frac{1}{\sqrt{p_{\alpha_1\ldots\alpha_k}}} A^{\alpha_1}_{\text{in}} \ldots A^{\alpha_k}_{\text{in}}\Omega_{\text{in}}, \end{array}\right. \qquad (6) \qquad \left|\begin{array}{l} \Omega_{\text{out}} \\ \Phi^{\alpha}_{\text{out}} = A^{\alpha}_{\text{out}}\Omega_{\text{out}} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \Phi^{\alpha_1\ldots\alpha_k}_{\text{out}} = \frac{1}{\sqrt{p_{\alpha_1\ldots\alpha_k}}} A^{\alpha_1}_{\text{out}} \ldots A^{\alpha_k}_{\text{out}}\Omega_{\text{out}}. \end{array}\right. \qquad (7)$$

The abbreviated symbols mean

$$A^{\alpha}_{\text{out}} = A^{f_\alpha}_{\text{out}}(t), \qquad A^{\alpha}_{\text{in}} = A^{f_\alpha}_{\text{in}}(t), \qquad p_{\alpha_1\ldots\alpha_k} = n_1!\, n_2! \ldots n_r!,$$

where always n_i of the indices α are equal to each other.

The S-matrix is defined by

$$S_{(\alpha)(\beta)} = (\Phi^{(\alpha)}_{\text{out}}, \Phi^{(\beta)}_{\text{in}}) \tag{8}$$

where (α), (β) mean the index combinations

$$(\alpha) = \alpha_1, \ldots \alpha_k, \qquad (\beta) = \beta_1, \ldots, \beta_l$$

The unitarity of the S-matrix follows directly, as also the relation

$$A_{\text{out}} = S^\dagger A_{\text{in}} S. \tag{9}$$

Occasionally we shall speak of "mixed matrix elements" of field operators or their products. By this we mean expressions of the form

$$(\Phi^{(\alpha)}_{\text{out}}, A(x)\Phi^{(\beta)}_{\text{in}}),$$

which on the left side refers to a state of the orthonormal system (7) and on the right refers to a state of system (6).

From the asymptotic condition and the invariance principle, the existence of an energy-momentum vector P_μ follows with the properties

$$-i\,[P_\mu, A(x)] = \frac{\partial A(x)}{\partial x_\mu}, \qquad -i\,[P_\mu, A_{\text{in}}(x)] = \frac{\partial A_{\text{in}}(x)}{\partial x_\mu}, \qquad -i\,[P_\mu, A_{\text{out}}(x)] = \frac{\partial A_{\text{out}}(x)}{\partial x_\mu}, \tag{10}$$

and further

$$\Omega_{\text{in}} = \Omega_{\text{out}} = \Omega \tag{11}$$

(the arbitrary phase factor is set equal to unity). P_μ fulfills, on the basis of invariance

$$P_\mu \Omega = 0.$$

The definition of Φ^α_{in} or Φ^α_{out} shows that $(\Omega, A(x)\Phi^\alpha_{\substack{\text{in}\\ \text{out}}})$ is a solution of the Klein-Gordon equation. With the asymptotic condition, it follows that

$$\left\{ \begin{array}{l} (\Omega, A(x)\Phi^\alpha_{\text{in}}) = (\Omega, A_{\text{in}}(x)\Phi^\alpha_{\text{in}}) = f_\alpha(x) \; {}^{(8)} \\ (\Omega, A(x)\Phi^\alpha_{\text{out}}) = (\Omega, A_{\text{out}}(x)\Phi^\alpha_{\text{out}}) = f_\alpha(x) \end{array} \right. \tag{12}$$

and from it $\Phi^\alpha_{\text{in}} = \Phi^\alpha_{\text{out}}$. A further consequence of the asymptotic condition is the vanishing of the vacuum expectation value of $A(x)$:

$$(\Omega, A(x)\Omega) = (\Omega, A_{\text{in}}(x)\Omega) = 0\,. \tag{13}$$

III. THE REDUCTION FORMULA

After establishing the foundation of the theory, one should ask which type of fields satisfy the above assumptions and how can the S-matrix and field operators be calculated for a given case. The general operator form of the postulates seems little suited for answering these questions. Therefore, it shall be the task in this section to develop a system of equations equivalent to the three postulates so that an analytical treatment can be developed. We shall use the condition that as a result of the asymptotic condition the matrix elements of the field operators and the S-matrix may be reduced completely to the vacuum-τ-functions. (By this we mean the vacuum expectation values of T-products of field operators.) This means it is enough to limit oneself to the vacuum-τ-functions. However, these can, in principle, be calculated from an infinite system of equations (system A) that we will set up toward the end of this section. The derivation of the system A follows from the asymptotic condition. The invariance principle and the causality conditions produce the invariance of the vacuum-τ-functions. Thus the task of determining and calculating all fields $A(x)$ which satisfy three basic postulates of invariance, causality, and the asymptotic condition, reduces to the discussion of the invariant solutions of system A.[9]

This "reduction" of the S-matrix and field operators is only a special case of the general reduction formula which states that the mixed matrix elements of an arbitrary T-product of field operators may be expressed in an elementary way entirely by the vacuum expectation value of the T-products. The reduction formula is a result of the asymptotic condition. It will be formulated and proven in complete generality in a functional way in the appendix. Here we prove it only insofar as it is necessary for the reduction of the S-matrix and for the determination of the system A.

The asymptotic condition should be assumed in the form (5). Further, the relation

$$\Omega_{\text{in}} = \Omega_{\text{out}} = \Omega\,.$$

should be valid. We first prove the special case

$$(\Omega,\ T(x_1,\ldots,x_n)\Phi^{\alpha}_{\mathrm{in}}) = -i\int K_y\tau(x_1,\ldots,x_n,y)f_{\alpha}(y)\,d^4y \tag{14}$$

of the reduction formula where, for abbreviation, we set:

$$T(x_1,\ldots,x_n) = T\{A(x_1)\ldots A(x_n)\};$$

$$K_y = \Box_y - m^2$$

and

$$\tau(x_1,\ldots,x_n) = (\Omega,\ T(x_1,\ldots,x_n)\Omega) \tag{15}$$

means the vacuum-τ-function of $\underline{n}$ arguments. From the definition of $\Phi^{\alpha}_{\mathrm{in}}$ it follows that:

$$(\Omega,\ T(x_1,\ldots,x_n)\Phi^{\alpha}_{\mathrm{in}}) = (\bar{T}(x_1,\ldots,x_n)\Omega,\ A^{\alpha}_{\mathrm{in}}\Omega)\ .$$

Application of the asymptotic condition to the right side yields:

$$\begin{aligned}(\Omega,\ T(x_1,\ldots,x_n)\Phi^{\alpha}_{\mathrm{in}}) &= \lim_{y_0\to-\infty}(\bar{T}(x_1,\ldots,x_n)\Omega,\ A^{\alpha}(y_0)\Omega)\\ &= i\lim_{y_0\to-\infty}\int(\Omega,\ T(x_1,\ldots,x_n,y)\overleftrightarrow{\frac{\partial}{\partial y_0}}f_{\alpha}(y)\,d^3y\\ &= -i\int\frac{\partial}{\partial y_0}\left\{\tau_0(x_1,\ldots,x_n,y)\overleftrightarrow{\frac{\partial}{\partial y_0}}f_{\alpha}(y)\,d^4y\right\},\end{aligned}$$

with the abbreviations:

$$A^{\alpha}(t) = A^{f_{\alpha}}(t)\,,\qquad f(x)\overleftrightarrow{\frac{\partial}{\partial x}}g(x) = f(x)\frac{\partial g(x)}{\partial x_0} - g(x)\frac{\partial f(x)}{\partial x_0}\,,$$

since the boundary term at $y_0 = +\infty$

$$i\lim_{y_0\to+\infty}\int(\Omega,\ T(x_1,\ldots,x_n,y)\Omega)\overleftrightarrow{\frac{\partial}{\partial y_0}}f_{\alpha}(y)\,d^3y = (\Omega,\ A^{\alpha}_{\mathrm{out}}T(x_1,\ldots,x_n)\Omega)\,,$$

and

$$(A^{\alpha}_{\mathrm{out}})^*\Omega = 0\,,$$

vanishes because of

$$i\int\frac{\partial}{\partial y_0}\left\{\tau_0(x_1,\ldots,x_n,y)\overleftrightarrow{\frac{\partial}{\partial y_0}}f_{\alpha}(y)\right\}d^4y\,,$$

gives, after several transformations that make use of the vanishing of the space-like boundary terms, the right side of equation (14) that was to be proven.

In an analogous way, one also proves the relation

$$(\Omega,\ T(x_1,\ldots,x_n)\Phi^{\alpha_1\ldots\alpha_{k+1}}_{\mathrm{in}}) = \int K_y(\Omega,\ T(x_1,\ldots,x_n,y)\Phi^{\alpha_1\ldots\alpha_k}_{\mathrm{in}})f_{\alpha_{k+1}}(y)\,d^4y\ . \tag{16}$$

From this follows, using only mathematical induction, the formula

$$\begin{aligned}(\Omega,\ T(x_1 \dots x_n)\Phi_{\text{in}}^{\alpha_1\dots\alpha_k}) &= \\ &= (-i)^k \int K_{y_1} \dots K_{y_k} \tau(x_1 \dots x_n y_1 \dots y_k) f_{\alpha_1}(y_1) \dots f_{\alpha_k}(y_k)\, \mathrm{d}^4y_1 \dots \mathrm{d}^4y_k\,. \end{aligned} \tag{17}$$

This is the reduction formula for such matrix elements of T-products which have a vacuum state on the left. Accordingly, one can evaluate the matrix elements of T-products taken between the vacuum and an arbitrary state of the orthonormal system (6) from the vacuum-τ-functions. By transforming to the conjugate complex quantity, one obtains the reduction formula for T-products matrix elements which have the vacuum state on the right.

We now turn to the reduction of the S-matrix. Our purpose is to reduce the matrix element

$$(\Phi_{\text{out}}^{(\alpha)},\ \Phi_{\text{in}}^{(\beta)}) \qquad (\alpha) = \alpha_1, \dots, \alpha_k\,,\quad (\beta) = \beta_1, \dots, \beta_l$$

to vacuum-τ-functions, first of all, for the case where none of the indices α_i agree with any of the indices β_i. The generalization of equation (16) then reads:

$$(\Phi_{\text{out}}^{(\alpha)},\ T(x_1 \dots x_n)\Phi_{\text{in}}^{(\beta)}) = -i \int K_\xi (\Phi_{\text{out}}^{(\alpha)},\ T(x_1 \dots x_n \xi)\Phi_{\text{in}}^{\beta_1 \dots \beta_{l-1}}) f_{\beta_l}(\xi)\, \mathrm{d}^4\xi\,,$$

$$(\Phi_{\text{out}}^{(\alpha)},\ T(x_1 \dots x_n)\Phi_{\text{in}}^{(\beta)}) = -i \int K_\eta (\Phi_{\text{out}}^{\alpha_1 \dots \alpha_{k-1}},\ T(x_1 \dots x_n \eta)\Phi_{\text{in}}^{(\beta)}) f_{\alpha_k}^{*}(\eta)\, \mathrm{d}^4\eta.$$

By mathematical induction, it follows that:

$$\begin{aligned}(\Phi_{\text{out}}^{(\alpha)},\ \Phi_{\text{in}}^{(\beta)}) = (-i)^{k+l} \int K_{\xi_1} \dots K_{\xi_k} K_{\eta_1} \dots K_{\eta_l} \tau(\xi_1 \dots \xi_k \eta_1 \dots \eta_l) f^*(\xi_1) \dots f^*(\xi_k)\cdot \\ \cdot f(\eta_1) \dots f(\eta_l)\, \mathrm{d}^4\xi_1 \dots \mathrm{d}^4\xi_k\, \mathrm{d}^4\eta_1 \dots \mathrm{d}^4\eta_l\,. \end{aligned}$$

From this, one easily derives a normal form for the S-operator which is defined by

$$(\Phi_{\text{out}}^{(\alpha)},\ \Phi_{\text{in}}^{(\beta)}) = (\Phi_{\text{in}}^{(\alpha)},\ S\Phi_{\text{in}}^{(\beta)}) \tag{18}$$

then

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int K_1 \dots K_n \varphi(x_1 \dots x_n) : A_{\text{in}}(x_1) \dots A_{\text{in}}(x_n) : d^4x_1 \dots d^4x_n\,. \tag{19}$$

The functions $\varphi(x_1, \dots, x_n)$ mean the vacuum-φ-functions[10] which are formed according to Wick's rule with the contraction function Δ_F. Equation (19) is generally valid, independent of the assumption made in the beginning about the indices. It is to be noted that for the integration of the integral in momentum space, the vacuum-τ-functions must be known only for values that lie on the mass hyperboloid. Only the vacuum-τ-functions which are so limited enter into the evaluation of the S-matrix.

Finally, we derive the infinite system of equations A that will be used in the determination of the vacuum-τ-functions. We start from the simple operator identity

$$T\{A(x_1)\dots A(x_n)\} = \sum_v \theta(x_1 - x_n)\dots\theta(x_{n-1} - x_n)T\{A(x_1)\dots A(x_{n-1})\}A(x_n)$$

where $\sum\limits_V$ means summation over all n − 1 interchanges of a coordinate x_i with x_n. Formation of the vacuum expectation value and the decomposition of the right side gives:

$$(\Omega, T(x_1,\dots,x_n)\Omega) = \sum_v \sum_{(\alpha)} \theta(x_1 - x_n)\dots\theta(x_{n-1} - x_n)(\Omega, T(x_1\dots x_n)\Phi_{\text{in}}^{(\alpha)})(\Phi_{\text{in}}^{(\alpha)}, A(x_n)\Omega) =$$

$$= \sum_v \sum_{k=1}^{\infty} \sum_{\alpha_1\dots\alpha_k} \theta(x_1 - x_n)\dots\theta(x_{n-1} - x_n)(\Omega,\ T(x_1\dots x_{n-1})\Phi_{\text{in}}^{\alpha_1\dots\alpha_k})(\Phi_{\text{in}}^{\alpha_1\dots\alpha_k}, A(x_n)\Omega)\,.$$

If now we also express the matrix elements on the right according to the reduction formula (17) with the vacuum-τ-functions, then we obtain the desired system of equations involving only the vacuum-τ-functions:[11]

$$\tau(x_1,\dots,x_n) = \sum_v \sum_{k=1}^{\infty} \theta(x_1 - x_n)\dots\theta(x_{n-1} - x_n)i^k \int d^4\xi_1\dots d^4\xi_k\, d^4\eta_1\dots d^4\eta_k$$

$$K_{\xi_1}\dots K_{\xi_k}\tau(x_1\dots x_{n-1}\xi_1\dots\xi_r)\Delta^+(\xi_1 - \eta_1)\dots\Delta^+(\xi_k - \eta_k)K_{\eta_1}\dots K_{\eta_k}\tau^*(x_n\eta_1\dots\eta_k). \tag{A}$$

From the invariance principle and the causality condition, it follows that the vacuum-τ-functions are invariant functions. In order to obtain a general idea of all fields satisfying the three postulates of invariance, causality and asymptotic conditions, it is therefore sufficient to investigate the invariant solutions of system A.

We mention another system of equations A′ for the τ-functions which can be obtained in similar fashion beginning with the operator identity

$$T\{A(x_1)\dots A(x_n)\} = \sum_v \theta(x_n - x_1)\dots\theta x_n - x_{n-1})A(x_n)T\{A(x_1)\dots A(x_{n-1})\}$$

The system A′ is equivalent to system A.

Another system of equations of vacuum-τ-functions which appears appropriate for practical applications, can be obtained by insertion of the reduction formula into the following identity:

$$\begin{aligned}\sum_{(\beta)} (\Phi_{\text{in}}^{(\alpha)}, \Phi_{\text{out}}^{(\beta)})(\Phi_{\text{out}}^{(\beta)}, TA(x_1)A(x_2)\Phi_{\text{in}}^{(\gamma)}) &= \\ = \theta(x_1 - x_2)\sum_{(\beta)} (\Phi_{\text{in}}^{(\alpha)}A(x_1)\Phi_{\text{out}}^{(\beta)})(\Phi_{\text{out}}^{(\beta)}, A(x_2)\Phi_{\text{in}}^{(\gamma)}) &+ \text{Symm.}\end{aligned} \tag{20}$$

This system is also equivalent to system A.

IV. DISCUSSION OF SYSTEM OF EQUATIONS A

Due to the few conditions that were required for the derivation of system A, one would expect that these equations possess a great number of solutions. This section will give a survey of the invariant solutions and, in particular, show that one can select definite solutions by fixing boundary conditions. For this we limit ourselves to a perturbation theory discussion of the equations.

First we note that, of course, the known τ-functions of a free field form an exact solution of system A. They are

$$\tau(x_1, x_2) = \Delta_F(x_1 - x_2)\,; \qquad \tau(x_1, \ldots, x_{2n+1}) = 0\,,$$

$$\tau(x_1, \ldots, x_{2n}) = \frac{1}{n} \sum_{\mu<\nu} \Delta_F(x_\mu - x_\nu)\tau(x_1 \ldots x_{\mu-1}, x_{\mu+1} \ldots x_{\nu-1}, x_{\nu+1} \ldots x_{2n})\,. \tag{21}$$

For further treatment, it is useful to transform to the functions $\varphi(x_1 \ldots x_n)$ already introduced in Section III. We can easily transform this system into a system of equations for the φ-functions [see Appendix) (A-11]. We will now determine the φ-functions in the first approximation to the free fields. It will be shown that this calculation has a meaning beyond the first approximation and can serve as a survey of the possible solutions of system A for an arbitrary approximation. Let

$$\varphi(x_1, \ldots, x_n) = g\varphi^{(1)} + g^2\varphi^{(2)} + \ldots\,.$$

We assume that in the first approximation (~g), only a certain function $\varphi(x_1 \ldots x_n)$ is different from zero, that is, deviates from a free field[12]. From (A) it follows then for $\varphi^{(1)}(x_1, \ldots, x_n)$

$$\varphi^{(1)}(x_1, \ldots x_n) =$$

$$= \sum_v \theta(x_1 - x_n) \ldots \theta(x_{n-1} - x_n) \left\{ \int K_{\xi_n} \varphi^{(1)}(x_1, \ldots, x_{n-1}, \xi_n) \Delta^+(\xi_n - x_n)\, d^4\xi_n + \right. \tag{22}$$

$$\left. + i^{2(n-1)} \int \Delta^+(x_1 - \xi_1) \ldots \Delta^+(x_{n-1} - \xi_{n-1}) K_{\xi_1} \ldots K_{\xi_{n-1}} \varphi^{(1)*}(x_n, \xi_1, \ldots, \xi_{n-1})\, d^4\xi_1 \ldots d^4\xi_{n-1} \right\}.$$

It is convenient not to solve this equation for $\varphi(x_1 \ldots x_n)$, but for $\omega(x_1 \ldots x_n)$, with

$$\varphi(x_1 \ldots x_n) = \int \Delta_F(x_1 - \xi_1) \ldots \Delta_F(x_n - \xi_n) \omega_n(\xi_1, \ldots, \xi_n)\, d^4\xi_1 \ldots d^4\xi_n\,. \tag{23}$$

Equation (22) then becomes

$$\int \Delta_F(x_1 - \xi_1) \ldots \Delta_F(x_n - \xi_n) \omega^{(1)}(\xi_1, \ldots, \xi_n)\, d^4\xi_1 \ldots d^4\xi_n =$$

$$= \sum_v \theta(x_1 - x_n) \ldots \theta(x_{n-1} - x_n) \cdot$$

$$\cdot \int \{\Delta_F(x_1 - \xi_1) \ldots \Delta_F(x_{n-1} - \xi_{n-1})(-i)\Delta^-(x_n - \xi_n)\omega^{(1)}(\xi_1, \ldots, \xi_n) + \tag{22'}$$

$$+ i\Delta^+(x_1 - \xi_1) \ldots i\Delta^+(x_{n-1} - \xi_{n-1})\Delta_F(x_n - \xi_n)\omega^{(1)*}(\xi_1 \ldots \xi_n)\}\, d^4\xi_1 \ldots d^4\xi_n\,.$$

By applying the relations

$$-i\Delta^-(x) = \Delta_F(x) + i\Delta_R(x)\,; \qquad i\Delta^+(x) = \bar{\Delta}_F(x) - i\Delta_R(x)$$

it follows from (22′)

$$\sum_{r} \theta(x_1 - x_n) \ldots \theta(x_{n-1} - x_n) \cdot$$
$$\cdot \int \{\Delta_F(x_1 - \xi_1) \ldots \Delta_F(x_{n-1} - \xi_{n-1}) \Delta_R(x_n - \xi_n) \omega^{(1)}(\xi_1, \ldots \xi_n) + \tag{24}$$
$$+ i\Delta^+(x_1 - \xi_1) \ldots i\Delta^+(x_{n-1} - \xi_{n-1}) \Delta_R(x_n - \xi_n) \omega^{(1)*}(\xi_1, \ldots, \xi_n)\} \, d^4\xi_1 \ldots d^4\xi_n = 0 \, .$$

It will become apparent that the invariant functions $\omega^{(1)}$ are different from zero only when all the times $x_{10}, \ldots, x_{n0}$ are the same. Due to the invariance it follows from this that also the space components of the arguments must be equal, i.e., the function $\omega^{(1)}$ consists of products of four-dimensional δ-functions and their derivatives.

We prove this statement first for $\omega^{(1)}(x_1, x_2, x_3)$ and then carry over the results for arbitrary n. For $\omega^{(1)}(x_1, x_2, x_3)$ (24) reads:

$$\theta(x_1 - x_3)\theta(x_2 - x_3) \int \{\Delta_F(x_1 - \xi_1)\Delta_F(x_2 - \xi_2)\Delta_R(x_3 - \xi_3)\omega^{(1)}(\xi_1, \xi_2, \xi_3) -$$
$$- \Delta^+(x_1 - \xi_1)\Delta^+(x_2 - \xi_2)\Delta_R(x_3 - \xi_3)\omega^{(1)*}(\xi_1\xi_2\xi_3)\} \, d^4\xi_1 \, d^4\xi_2 \, d^4\xi_3 + \tag{24'}$$
$$+ \theta(x_2 - x_1)\theta(x_3 - x_1) \{ \} + \theta(3, 1; 2) \{ \} = 0 \, .$$

By applying the Klein-Gordon operator it follows that

$$\omega^{(1)}(x_1, x_2, x_3) = 0 \quad \text{when} \quad \left| \begin{array}{l} x_{10}, x_{20} > x_{30} \\ x_{20}, x_{30} > x_{10} \\ x_{30}, x_{10} > x_{20} . \end{array} \right.$$

From the system of equations A′ mentioned in Section III, it similarly follows that:

$$\omega^{(1)}(x_1, x_2, x_3) = 0 \quad \text{when} \quad \left| \begin{array}{l} x_{10}, x_{20} < x_{30} \\ x_{20}, x_{30} < x_{10} \\ x_{30}, x_{10} < x_{20} . \end{array} \right.$$

These conditions show that in $\omega^{(1)}(x_1, x_2, x_3)$ all times must be equal. From the invariant functions which remain under this condition, there are three solutions to equation 24′. They are, in coordinate or momentum space, (g_{31}, g_{32}, g_{33} are real constants):

1) $\omega^{(1)} = ig_{31} \, \delta(x_1 - x_2) \, \delta(x_2 - x_3) \, ;$ bzw. $ig_{31} \cdot \delta(p_1 + p_2 + p_3) \, ,$

2) $\omega^{(1)} = ig_{32}(\Box_{x_1} + \Box_{x_2} + \Box_{x_3}) \, \delta(x_1 - x_2)\delta(x_2 - x_3);$
bzw. $-ig_{32}(p_1^2 + p_2^2 + p_3^2) \cdot \delta(p_1 + p_2 + p_3) \, ,$

3) $\omega^{(1)} = ig_{33}(\Box_1 \partial_{\mu_2} \partial_{\mu_3} + \Box_2 \partial_{\mu_3} \partial_{\mu_1} + \Box_3 \partial_{\mu_1} \partial_{\mu_2}) \, \delta(x_1 - x_2) \, \delta(x_2 - x_3) \, ;$
bzw. $ig_{33}(p_1^2(p_2p_3) + p_2^2(p_3p_1) + p_3^2(p_1p_2))\delta(p_1 + p_2 + p_3) \, .$

We verify this by taking as an example solution (1). Here (24′) reads:

$$\theta(1, 2; 3) \int \{\Delta_F(x_1 - \xi)\Delta_F(x_2 - \xi) + \Delta^+(x_1 - \xi)\Delta^+(x_2 - \xi)\} \cdot$$
$$\cdot \Delta_R(x_3 - \xi) \, d^4\xi + \theta(2, 3; 1) \ldots + \ldots = 0 \, .$$

This equation is satisfied since the Δ_F-functions can be replaced by $i\Delta^+$ because $x_{10}, x_{20} > x_{30} > \xi_0$.

There are certainly no arbitrarily high derivatives of δ-functions for solutions. Since, in general, by the differentiation of the Δ_F-functions, additional terms appear that destroy the equivalence of Δ_F and $i\Delta^+$, it can be shown that only three given solutions exist.

For $\omega^{(1)}(x_1, \ldots, x_n)$ one obtains similar results. One determines further that products of δ-functions and some derivatives of these functions satisfy equation (24). For $\omega^{(1)}(x_1, \ldots, x_n)$ there are $\underline{n}$ independent solutions in which arbitrary parameters $g_{n1}, \ldots, g_{nn}$ appear. In momentum space they are represented by symmetric polynomials in the variables $p_1, \ldots, p_n$, which are characterized by the fact that the sum of the powers of any two momenta is smaller than four.

The functions $\varphi^{(1)}$ (or $\omega^{(1)}$) obtained in such a way agree with the expressions which one obtains by using the customary formulation of field theories in the first approximation. To this, naturally, there correspond δ'-function theories with derivative coupling.

It is evident, then, that the equations discussed here are formally equivalent (i.e., with the exception of renormalization terms) to the conventional field equations. If, for example, we take the constants $g_{31}, g_{41}, \ldots, g_{n1}$ as different from zero, then this corresponds to a Lagrangian with the coupling term

$$L' = g'_{31}A^3(x) + g'_{41}A^4(x) + \ldots + g'_{n1}A^n(x) .$$

Thus the constants g_{i1} and g'_{i1} agree within the first approximation.

By means of the above solutions of the (homogeneous) equations for the $\varphi^{(1)}$-functions, the parameters $g_{n1}, \ldots, g_{nn}$ $(n = 3, 4, \ldots)$ are introduced, and they act as coupling constants. When these constants are given (therefore $\varphi^{(1)}$ is known) then one can attempt to determine the φ-functions to a higher approximation in a power series expansion in these constants with the help of system A. In the second approximation, one obtains for $\varphi^{(2)}$ an inhomogeneous equation in whose inhomogeneous terms the first approximation functions appear. The homogeneous part of the equation for $\varphi^{(2)}$ is identical to the equation already treated for $\varphi^{(1)}$. $\varphi^{(2)}$ is thus determined within solutions of this homogeneous equation. This state of affairs is true to arbitrary order of the perturbation calculation.

The necessary determination of the homogeneous solutions that appears through the perturbation calculation can now occur by putting boundary conditions on the solutions of this system A in which several of the asymptotic properties of the functions ω are given. In the one-parameter case (only $g_{n1} \neq 0$) this condition is: $\omega(x_1, \ldots, x_n)$; it should contain, (in every approximation) for large momenta, a constant germ g_{n1}; it should not contain any terms of the form $g_{n2}(p_1^2 + \ldots p_n^2)$; $g_{n3}(p_1^2 p_2 p_3 + \ldots)$, etc., that correspond to other solutions of the homogeneous equation for $\omega^{(1)}(x_1 \ldots x_n)$. All other functions ω should asymptotically contain no terms that are constant or that represent polynomials corresponding to the homogeneous equations. In general, one is given the coupling constants and with them the asymptotic behavior of the function ω corresponding to these constants.[13]

As is known, among the one-parameter theories, there are exactly two that are renormalizable in the framework of the usual formulation For these, g_{31} or g_{41} is different from zero. (Coupling $\sim A^3$ or $\sim A^4$, respectively.) With these boundary conditions, the system A has unambiguously determined solutions in every approximation which correspond to the renormalized expressions.[14]

The assertions made in this section refer, of course, only to the perturbation calculation (of an arbitrary order). One of the advantages of system A, that a discussion of renormalizable theories should be possible without being restricted to the perturbation treatment, has so far not been introduced. However, a perturbation treatment of system A should be more consistent than the usual method, since here we deal only with divergence-free equations. We do not investigate the uniqueness of the perturbation calculation because the model character of the scalar field treated here does not make this significant.

APPENDIX

FUNCTIONAL FORMULATION

The ordered operator[15]

$$T \exp \left[i \int_{t_1}^{t_2} A(x) J(x) \, \mathrm{d}x \right| \quad \mathfrak{T}\{t_2, t_1; J\}$$

represented as a functional of the space-time function J(x) is determined by the integral equation

$$\mathfrak{T}\{t_2, t_1; J\} = 1 + i \int_{t}^{t_2} \mathfrak{T}\{t_2, x_0; J\} J(x) A(x) \, \mathrm{d}x \tag{A.1}$$

For abbreviation we set:

$$\mathfrak{T}\{+\infty, -\infty; J\} \equiv \mathfrak{T}\{J\} \quad \text{and} \quad \langle | \mathfrak{T}\{J\} | \rangle \equiv \mathfrak{T}_0\{J\},$$

where for clarity we employ the Dirac notation $|\rangle$ for the vacuum.

By virtue of

$$\mathfrak{T}_0\{J\} = \sum_{n=0}^{\infty} \frac{i^n \tau_n\{J\}}{n!},$$

with

$$\tau_n\{J\} \equiv \int \dots \int \mathrm{d}x_1 \dots \mathrm{d}x_n \, \tau(x_1 \dots x_n) J(x_1) \dots J(x_n),$$

this functional is the generating functional of the vacuum-τ-function (equation (15)), which can be obtained from $\mathcal{T}_0\{J\}$ by functional differentiation and setting J = 0. We obtain the operator

$$\mathcal{T}\{J\} = \sum_{(\alpha)} \sum_{(\beta)} |(\alpha)\rangle\langle(\alpha)|\mathcal{T}\{J\}|(\beta)\rangle\langle(\beta)|$$

in which we insert for $|(\alpha)|\rangle$ the orthonormal system of outgoing asymptotical free particles and for $|(\beta)|\rangle$ the incoming particles (equations (7) and (6)). Using the asymptotic condition (5) and the relation

$$\frac{\delta}{\delta J(x)} \mathcal{T}\{J\} = i\mathcal{T}\{+\infty, x_0; J\} A(x) \mathcal{T}\{x_0, -\infty; J\} \tag{A.2}$$

which follows from (A.1), we obtain

$$\begin{aligned} \langle(\alpha)|\mathcal{T}\{J\}|(\beta)\rangle &= \langle(\alpha)|\mathcal{T}\{J\} \frac{1}{\sqrt{P_{(\beta)}}} \prod_{\nu=1}^{n} \left[i \int A_{\text{in}}^{(-)}(x) \overleftrightarrow{\partial_0} f_{\beta_\nu}(x)\, d_3x \right] |\rangle = \\ &= \frac{1}{\sqrt{P_{(\beta)}}} \prod_{\nu=1}^{n} \left[\lim_{x_0 \to -\infty} \int \frac{\delta}{\delta J(x)} \overleftrightarrow{\partial_0} f_{\beta_\nu}(x)\, d_3x \right| \langle(\alpha)|\mathcal{T}\{J\}\ \rangle \end{aligned} \tag{A.3}$$

with the abbreviations

$$f(x) \overleftrightarrow{\partial_0} g(x) \equiv - g(x) \frac{\partial}{\partial x_0} f(x) + f(x) \frac{\partial}{\partial x_0} g(x).$$

Corresponding equations are obtained for $\varphi_{(\alpha)}$. With the completeness relation we obtain, by summing over all initial and final states:

$$\begin{aligned} \mathcal{T}\{J\} &= \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{1}{n!\, n'!} \left[\sum_{\alpha} i \int A_{\text{out}}^{(-)}(x) \overleftrightarrow{\partial_0} f_\alpha(x)\, d_3x \cdot \lim_{x_0 \to +\infty} \int f_\alpha^*(x') \overleftrightarrow{\partial_0'} \frac{\delta}{\delta J(x')}\, d_3x' \right]^n |\rangle \cdot \\ &\cdot \langle| \left[\sum_{\beta} \lim_{y_0 \to -\infty} \int \frac{\delta}{\delta J(y)} \overleftrightarrow{\partial_0} f_\beta(y)\, d_3y \cdot i \int f_\beta^*(y') \overleftrightarrow{\partial_0'} A_{\text{in}}^{(+)}(y')\, d_3y' \right]^{n'} \mathcal{T}_0\{J\} = \\ &= \exp\left[\lim_{x_0 \to \infty} \int A_{\text{out}}^{(-)}(x) \overleftrightarrow{\partial_0} \frac{\delta}{\delta J(x)}\, d_3x \right] |\rangle\langle| \exp\left[\lim_{y_0 \to -\infty} \int \frac{\delta}{\delta J(y)} \overleftrightarrow{\partial_0} A_{\text{in}}^{(+)}(y)\, d_3y \right] \mathcal{T}_0\{J\}. \end{aligned} \tag{A.4}$$

In order to be able to substitute volume integrals for surface integrals, the integrals over the opposite surface must vanish in each case. In (A.4) this is not yet the case. (Since (A.3) follows from (A.2) and the asymptotic condition only when J(x) vanishes sufficiently strongly on the boundary; but then further functional derivatives may not be arbitrarily taken in the neighborhood of the boundary, whatever the reason is for non-vanishing of the surface integral introduced by transforming (A.4)). Hence, by

$$\underset{0}{\mathcal{T}}\{J\} = \exp\left[-\frac{J \Delta_F J}{2} \right] \mathcal{S}_0\{J\}, \tag{A.5}$$

with

$$\mathcal{S}_0\{J\} = \sum_{n=0}^{\infty} \frac{i^n \varphi_n\{J\}}{n!} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \dots \int dx_1 \dots dx_n\, \varphi(x_1 \dots x_n) J(x_1) \dots J(x_n)$$

we obtain the generating functional of the vacuum-φ-functions.[10] For abbreviation we assume here and in later similar expressions, with matrix multiplication assumed, the following

$$J\Delta_r J \equiv \iint J(x)\Delta_r(x-x')J(x')\,\mathrm{d}x\,\mathrm{d}x'$$

$$A_{\mathrm{in}}\boldsymbol{K}\frac{\delta}{\delta J} \equiv \int A_{\mathrm{in}}(x)(\Box_x - m^2)\frac{\delta}{\delta J(x)}\,\mathrm{d}x \qquad \text{etc.}$$

Substitution of (A.5) in (A.4) gives

$$\mathcal{T}\{J\} = \exp\left[\int\limits_{x_0=\infty} A^{(-)}_{\mathrm{out}}(x)\overleftrightarrow{\partial_0}\frac{\delta}{\delta J(x)}\,\mathrm{d}_3x\right]\exp\left[\frac{-J\Delta_r J}{2}\right]|\rangle\cdot$$

$$\cdot\langle|\exp\left[-\int\limits_{y_0'=-\infty} J(y)\Delta_r(y-y')\overleftrightarrow{\partial_0'}A^{(+)}_{\mathrm{in}}(y')\,\mathrm{d}y\,\mathrm{d}_3y'\right]\cdot$$

$$\cdot\exp\left[\int\limits_{z_0=-\infty}\frac{\delta}{\delta J(z)}\overleftrightarrow{\partial_0}A^{(+)}_{\mathrm{in}}(z)\,\mathrm{d}_3z\right]\mathcal{S}_0\{J\} =$$

$$= \exp\left[-\frac{J\Delta_r J}{2}\right]\exp\left[\int\limits_{x_0'=\infty} J(x)\Delta_r(x-x')\overleftrightarrow{\partial_0'}A^{(-)}_{\mathrm{out}}(x')\,\mathrm{d}_3x'\,\mathrm{d}x\right.\cdot$$

$$\cdot\exp\left[\int\limits_{y_0=\infty} A^{(-)}_{\mathrm{out}}(y)\overleftrightarrow{\partial_0}\frac{\delta}{\delta J(y)}\,\mathrm{d}_3y\right]S^{+}\cdot\exp\left[-\int\limits_{z_0=-\infty} J'(z)\Delta_r(z-z')\overleftrightarrow{\partial_0'}A^{(+)}_{\mathrm{in}}\,\mathrm{d}z\,\mathrm{d}_3z'\right]\cdot$$

$$\cdot\exp\left[\int\limits_{z_0''=-\infty}\frac{\delta}{\delta J(z'')}\overleftrightarrow{\partial_0''}A^{(+)}_{\mathrm{in}}(z'')\,\mathrm{d}_3z''\right]\varphi_0\{J\}\cdot\Big|_{J'=J}$$

or

$$\mathcal{S}\mathcal{T}\{J\} = \exp\left[-\frac{J\Delta_r J}{2}\right]:\exp[iJA_{\mathrm{in}}]\exp\left[-A_{\mathrm{in}}\boldsymbol{K}\frac{\delta}{\delta J}\right]:\mathcal{S}_0\{J\} \qquad \text{(A.6a)}$$

and

$$\mathcal{T}\{J\}\mathcal{S} = \exp\left[-\frac{J\Delta_r J}{2}\right]:\exp[iJA_{\mathrm{out}}]\exp\left[-A_{\mathrm{out}}\boldsymbol{K}\frac{\delta}{\delta J}\right]:\mathcal{S}_0\{J\} \qquad \text{(A.6b)}$$

and with J = 0 finally

$$\mathcal{S} = :\mathcal{S}_0\{-A_{\mathrm{in}}\boldsymbol{K}\}: = :\mathcal{S}_0\{-A_{\mathrm{out}}\boldsymbol{K}\}: \qquad \text{(A.7)}$$

By this calculation, the series is, by Volterra's formula

$$\exp\left[J'\frac{\delta}{\delta J}\right]\mathcal{F}\{J\} = \mathcal{F}\{J+J'\}\exp\left[J'\frac{\delta}{\delta J}\right] \qquad \text{(A.8)}$$

($\mathcal{F}\{J\}$ is an arbitrary functional)

$$\int\limits_{x_0, x_0' = \mp\infty} A^{(\pm)}_{\substack{\mathrm{in}\\ \mathrm{out}}}(x)\overleftrightarrow{\partial_0}\Delta_r(x-x')\overleftrightarrow{\partial_0'}A^{(\pm)}_{\substack{\mathrm{in}\\ \mathrm{out}}}(x')\,\mathrm{d}_3x\,\mathrm{d}_3x' = 0\,,$$

$$S = :\exp\left[i\int A^{(-)}_{\mathrm{in}}(x)\overleftrightarrow{\partial_0}\,A^{(+)}_{\mathrm{out}}(x)\,\mathrm{d}_3x\right]:$$

and

$$A_{\mathrm{out}}(x) = S^{+}A_{\mathrm{in}}(x)S\,, \qquad A_{\mathrm{in}}(x) = SA_{\mathrm{out}}(x)S^{+}$$

is employed. In the formula finally used for the S-matrix, the operator $|\rangle\langle|$ is to be thought of as inserted in place of the unit operator between $A^{(-)}_{\mathrm{in}}$ and $A^{(+)}_{\mathrm{out}}$. The substitution that takes place in (A.6), of the surface integral by a volume integral, which in (A.4) was not allowed, is allowed here.

(A.7) is the normal form of the S-matrix (19), (A.6a,b) are the general reduction formulas mentioned in section III through which the matrix elements of the ordered operators are reduced to vacuum expectation values.

With

$$\widehat{J}(x', x) \equiv J(x')\theta(x'-x) \quad \text{and} \quad \overline{\mathcal{T}}\{J\} = (\mathcal{T}\{J\})^{+}$$

(A.1) can also be written as

$$\mathcal{T}\{J\} = 1 - \int\limits_{x_0=-\infty}^{+\infty} J(x)\mathcal{T}\{\widehat{J}\}\,\frac{\delta}{\delta J''(x)}\,\overline{\mathcal{T}}\{J''\}\,\mathrm{d}x\,\Big|_{J''=0}$$

Formation of the vacuum expectation value and use of (A.4) gives the functional form of system A:

$$\begin{aligned}
\mathcal{T}_0\{J\} &= 1 - \int\limits_{x_0=-\infty}^{\infty} J(x)\langle|\exp\left[\int\limits_{x_0'=-\infty}\frac{\delta}{\delta J'(x')}\overleftrightarrow{\partial_0'}A^{(+)}_{\mathrm{in}}(x')\mathrm{d}_3x'\right]\cdot\\
&\quad\cdot\exp\left[-\int\limits_{x_0''=-\infty}A^{(-)}_{\mathrm{in}}(x'')\overleftrightarrow{\partial_0'}\frac{\delta}{\delta J''(x'')}\,\mathrm{d}_3x''\right]|\rangle\cdot\frac{\delta}{\delta J''(x)}\,\mathcal{T}_0\{J'\}\overline{\mathcal{T}}_0\{J''\}\,\mathrm{d}x\,\Big|_{\substack{J'=\widehat{J}\\ J''=0}} =\\
&= 1 - \int\limits_{x_0=-\infty}^{\infty} J(x)\,\frac{\delta}{\delta J''(x)}\exp\left[-\iint\limits_{x_0', x_0''=-\infty}\frac{\delta}{\delta J'(x')}\overleftrightarrow{\partial_0'}i\Delta^{(+)}(x'-x'')\overleftrightarrow{\partial_0''}\frac{\delta}{\delta J''(x'')}\,\mathrm{d}_3x'\,\mathrm{d}_3x''\right]\cdot\\
&\quad\cdot\mathcal{T}_0\{J'\}\overline{\mathcal{T}}_0\{J''\}\,\mathrm{d}x\,\Big|_{\substack{J'=\widehat{J}\\ J''=0}} =\\
&= 1 - \int\limits_{x_0=-\infty}^{\infty} J(x)\,\frac{\delta}{\delta J''(x)}\exp\left[\frac{\delta}{\delta J'}\,\overleftarrow{K}i\Delta^{(+)}\overrightarrow{K}\,\frac{\delta}{\delta J''}\right]\mathcal{T}_0\{J'\}\overline{\mathcal{T}}_0\{J''\}\,\mathrm{d}x\,\Big|_{\substack{J'=\widehat{J}\\ J''=0}}.
\end{aligned} \tag{A.9}$$

From this it follows, on account of

$$\mathcal{T}_0\{J\} = 1 - \int\limits_{-\infty}^{\infty}\mathrm{d}x_0\,\frac{\mathrm{d}}{\mathrm{d}x_0}\,\mathcal{T}_0\{\widehat{J}\} = 1 + \int\limits_{x_0=-\infty}^{\infty} J(x)\,\frac{\delta}{\delta J'(x)}\,\mathcal{T}_0\{J'\}\,\mathrm{d}x\,\Big|_{J'=\widehat{J}} \tag{A.10}$$

and after insertion of (A.5) and repeated use of (A.8) that

$$\int_{x_0=-\infty}^{\infty} J(x) \exp\left[-\frac{\widehat{J}\Delta_r\widehat{J}}{2}\right]\left(-\int \Delta_r(x-x')\widehat{J}(x',x)\,\mathrm{d}x' + \frac{\delta}{\delta J'(x)}\right)\mathcal{S}_0\{J'\}\,\mathrm{d}x\,\Big|_{J'=\widehat{J}} =$$

$$= -\int_{x_0=-\infty}^{\infty} J(x) \exp\left[-\frac{\widehat{J}\Delta_F\widehat{J}}{2}\right]\left[\int \Delta^{(-)}(x-x')\left(-i\widehat{J}(x',x) + \boldsymbol{K}_{x'}\frac{\delta}{\delta J'(x')}\right)\mathrm{d}x' + \frac{\delta}{\delta J''(x)}\right]\cdot$$

$$\cdot \exp\left[\frac{\delta}{\delta J'}\overleftarrow{K} i\Delta^{+}\overrightarrow{K}\frac{\delta}{\delta J''}\right]\mathcal{S}_0\{J'\}\bar{\mathcal{S}}_0\{J''\}\,\mathrm{d}x\Big|_{\substack{J'=\widehat{J}\\ J''=0}} .$$

It can be shown by a series of expansion in powers of J that a similar equation holds in which the factor $\exp[-J\Delta_F J/2]$ is omitted from both sides. With this, we finally obtain by using $\mathcal{S}_0\{J\}$, a relation corresponding to (A.10), a condensation of system A for φ-functions:

$$\mathcal{S}_1\{J\} = 1 - \iint J(x)\Delta^{(-)}(x-x')\boldsymbol{K}_{x'}\frac{\delta}{\delta J'(x')}\mathcal{S}_0\{J'\}\,\mathrm{d}x\,\mathrm{d}x'\Big|_{J'(x')=\widehat{J}(x',x)} -$$

$$- \int J(x)\frac{\delta}{\delta J''(x)}\exp\left[J'\Delta^{(+)}\boldsymbol{K}\frac{\delta}{\delta J''}\right]\exp\left[\frac{\delta}{\delta J'}\overleftarrow{K} i\Delta^{(+)}\overrightarrow{K}\frac{\delta}{\delta J''}\right]\mathcal{S}_0\{J'\}\bar{\mathcal{S}}_0\{J''\}\,\mathrm{d}x\Big|_{\substack{J'=\widehat{J}\\ J''=0}} . \qquad (A.11)$$

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1. E. Wigner: Ann. of Math. 30, 149 (1939).
2. This form is also used, for example, in the article by M. Gel-Mann, M. L. Goldberger, and W. E. Thirring: Phys. Rev. 95, 1612 (1954).
3. E. C. G. Stuckelberg and G. Wanders: Acausalite de l'interaction non-locale, Manuscript 1954. [Helv. Phys. Acta. 27, 667 (1955)]
4. On the fundamental meaning of the asymptotic condition for the foundation of field theory, cf. R. Haag: On Quantum Field Theories, Manuscript 1954. [Dan. Mat. Fys. Medd, 29, paper 12]
5. Moreover, it is required that $A_{in}(x)$ and $A_{out}(x)$ give an irreducible representation of these commutation relations.
6. These conditions must be added to characterize the free fields since the canonical commutation relations permit non-equivalent representations, cf. K. O. Friedrichs: Math. Aspects of the Quantum Theory of Fields (New York, 1953). [Interscience Publishers, Inc., N.Y.C.]
7. Therefore, the time-independent operator $A^{f}_{in}(t)$, creates a particle with wave function f(x). In contrast to $A_{in}(x)$, these operators do not, by application to a normalized state, leave the Hilbert space.
8. These relations which follow here as a natural consequence of the asymptotic condition were used earlier by Källen for a determination of the renormalization constants. G. Källen: Helv. Phys. Acta. 25, 417 (1952).
9. One can also proceed from the commutator relation (1) to an analysis of possible field theories; cf. the work quoted in reference 4.

10. On the definition of τ- and φ-functions cf., i.e., the article W. Zimmermann: Suppl. Il Nuovo Cimento 11, 43 (1954), section 4, whose notation we have adopted.
11. In the usual formulation of field theory, the perturbation theory expansion of the vacuum-τ-functions satisfies the system A identically. The proof can be carried out analogously to the considerations sketched out in the note W. Zimmerman: Il Nuovo Cimento 11, 416 (1954).
12. This corresponds to the case where a coupling term with $\underline{n}$ field operators is treated in the Lagrangian formulation.
13. This condition corresponds in coordinate space to specifying the functions ω for equal arguments by means of δ-functions and their derivatives.
14. For non-renormalizable theories there are no solutions of system A within the perturbation calculation; e.g., the Δ_F'-function does not exist in the g^2-approximation for this kind of boundary condition.
15. For the properties of these operators introduced by Schwinger cf. K. Symanzik: Zeits. f. Naturf. 10a, 809 (1954) and the literature mentioned there.

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On the Formulation of Quantized Field Theories - II.

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Summary. — We discuss the concept of causal scattering matrices using retarded multiple commutators of field operators.

1. – Scattering matrix and field operators.

One of the main problems of relativistic quantum theory is to obtain information about scattering matrix elements. To this end it is customary to start from a quantized field theory; i.e. operator fields which obey certain equations and commutation relations. On the other hand one can try — following Heisenberg's original suggestion — to base a theory of interacting particles entirely on requirements imposed on the scattering matrix.

In what follows we intend to discuss the connection between such an S-matrix formalism and conventional quantized field theories.

Our aim is

a) to clarify some concepts employed in recent work; in particular to define rather precisely what is usually considered a « causal » scattering matrix;

b) to investigate some expressions which seem to be useful if one wants to derive causality conditions for a scattering matrix.

We begin with a pure S-matrix formalism (¹). No interacting field operators are considered but only an operator S which describes the scattering,

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(¹) W. Heisenberg: *Zeits. f. Phys.*, **120**, 513, 673 (1943).

production and annihilation of particles. Masses, spins etc. of all stable particles contained in the model are given parameters (2). First we require only that S describes a physical system in accordance with the general principles of quantum theory — with the exception of causality — and of special relativity. These concepts and their consequences for our problem have been discussed by HAAG (3) and this chapter is based on his work. The above principles imply: There exists a unitary representation of the inhomogenous Lorentz group in the Hilbert space of the state vectors.

The results of observations in the distant past or future may be characterized by states corresponding to non-interacting (incoming or outgoing) particles. The operator S gives a unitary mapping between initial and final states and therefore transition probabilities may be computed from S in the usual manner. For simplicity of notation we discuss a model containing only one kind of stable particle having mass m, spin 0 and no charge. For this case we give first the general form of S which follows from the mentioned principles. It is convenient to introduce the phase matrix η to take the unitarity of S into account.

$$S = \exp[i\eta] . \tag{1}$$

Then η is a hermitian operator which has the following structure:

$$\eta = \sum_{n=4}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \dots \mathrm{d}x_n h_n(x_1, \dots, x_n) : A_{\mathrm{in}}(x_1) \dots A_{\mathrm{in}}(x_n) : . \tag{2}$$

The operator field $A_{\mathrm{in}}(x)$ describes the incoming particles:

$$(\Box - m^2) A_{\mathrm{in}}(x) = 0 , \qquad [A_{\mathrm{in}}(x), A_{\mathrm{in}}(y)] = i\Delta(x-y) . \tag{3}$$

The double dots in (2) stand as usual for the Wick product. The functions $h_n(x_1, \dots, x_n)$ are c-numbers. They have to satisfy some general requirements. To discuss these it is useful to transform (2) into momentum space.

Let

$$A_{\mathrm{in}}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^4k \exp[ikx]\, \delta(k^2 + m^2) A_{\mathrm{in}}(k) , \tag{4}$$

$$h_n(x_1, \dots, x_n) = \tag{5}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}n}} \int \mathrm{d}k_1 \dots \mathrm{d}k_n \exp[-ik_1x_1 - \dots - ik_nx_n]\, \delta(k_1 + \dots + k_n) \tilde{h}_n(k_1, \dots, k_n) \quad (4) .$$

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(2) We do not attempt to distinguish between « elementary » and « bound » particles.

(3) R. HAAG: *Dan. Mat. Fys. Medd.*, 29, no. 12 (1955).

(4) The factor $\delta(k_1 + \dots + k_n)$ is of course a consequence of translation invariance and expresses energy-momentum conservation.

Then (2) becomes:

$$(2') \qquad \eta = \sum_{n=4}^{\infty} \frac{1}{n!} \int \mathrm{d}k_1 \dots \mathrm{d}k_n \, \delta(k_1 + \dots + k_n) \cdot$$

$$\cdot \tilde{h}_n(k_1, \dots, k_n) \, \delta(k_1^2 + m^2) \dots \delta(k_n^2 + m^2) : A_{\text{in}}(k_1) \dots A_{\text{in}}(k_n) : .$$

Thus only the values of $\tilde{h}_n(k_1, \dots, k_n)$ on the « energy shell » (mass hyperboloid) appear in this formula for η. Outside the energy shell the functions can be chosen in an arbitrary manner; i.e. $h_n(x_1, \dots, x_n)$ is by no means uniquely determined if η is given. On the energy shell the function $\tilde{h}_n(k_1, \dots, k_n)$ must have the following properties:

a) $\tilde{h}_n(k_1, \dots, k_n)$ is a symmetric function of $k_1, \dots, k_n$;

b) $\tilde{h}_n^*(k_1, \dots, k_n) = \tilde{h}_n(-k_1, \dots, -k_n)$ (this is equivalent to the hermiticity of η);

c) $\tilde{h}_n(k_1, \dots, k_n)$ is invariant with respect to the homogenous Lorentz group;

d) $\tilde{h}_n(k_1, \dots, k_n)$ contains no four-dimensional δ-functions. (Otherwise, not all observable quantities like cross-sections and their generalization for many-particle processes would be finite);

e) $\tilde{h}_n(k_1, \dots, k_n)$ is a continuous function of its invariant variables. (Continuity of scattering matrix elements);

f) $\tilde{h}_n(k_1, \dots, k_n) = \tilde{h}_n(-k_1, \dots, -k_n)$ (i.e. $\tilde{h}_n(k_1, \dots, k_n)$ real, because of (*b*)) follows if we require invariance against space-time inversion. If the theory is invariant under space or time inversion alone, then $\tilde{h}_n$ is invariant under $\boldsymbol{k}_i \to -\boldsymbol{k}_i$ or $k_{i0} \to -k_{i0}$.

All these are requirements on the energy shell. Since the extrapolation off the shell is so far arbitrary we can demand — and this will be done in what follows — that the functions satisfy the conditions *a*)-*f*) everywhere. Then we can also formulate these requirements in terms of $h_n(x_1, \dots, x_n)$ by Fourier transformation. The listed properties — which leave an enormous degree of arbitrariness — contain everything (as far as we know) that general physical principles, with the exception of causality, require in such a model.

We consider now the connection between such a scattering matrix and Lorentz invariant operator fields. In the framework of S-matrix formalism we have the relation

$$(6) \qquad A_{\text{out}}(x) = S^{\dagger} A_{\text{in}}(x) S \, .$$

We shall show: For any given S-matrix there exist (many!) invariant fields $A(x)$ which tend asymptotically for $t \to \pm \infty$ to the given fields $A_{\text{out}}(x)$ resp. $A_{\text{in}}(x)$.

A precise formulation of this asymptotic behavior is (5):

$$\lim_{t\to\pm\infty} (\Phi, A^f(t)\Psi) = (\Phi, A^f_{\substack{\text{out}\\ \text{in}}}\Psi)\,. \tag{7}$$

In this formula

$$A^f(t) = +\,i\int \mathrm{d}^3x \left\{A(x)\frac{\partial}{\partial x_0}f^*(x) - \frac{\partial}{\partial x_0}A(x)f^*(x)\right\}\ \text{(6)}\,,$$

(the operators A^f_{out}, A^f_{in} defined in the same manner are time independent) where $f(x)$ is an arbitrary normalizable solution of the Klein-Gordon equation.

We have, according to (6):

$$\begin{aligned} A_{\text{out}}(x) &= \exp[-i\eta]A_{\text{in}}(x)\exp[i\eta] \\ &= A_{\text{in}}(x) + i\int_0^1 \mathrm{d}\lambda\,\exp[-i\lambda\eta][A_{\text{in}}(x),\eta]\exp[i\lambda\eta] \\ &= A_{\text{in}}(x) - \int \mathrm{d}^4y\,\Delta(x-y)\int_0^1 \mathrm{d}\lambda\,\exp[-i\lambda\eta]\vartheta(y)\exp[i\lambda\eta]\,. \end{aligned} \tag{8}$$

The relation

$$[A_{\text{in}}(x),\eta] = i\int \Delta(x-y)\vartheta(y)\,\mathrm{d}^4y\,, \tag{9}$$

with

$$\vartheta(y) = \sum_{n=2}^{\infty}\frac{1}{n!}\int \mathrm{d}x_1\ldots\mathrm{d}x_n h_{n+1}(y, x_1, \ldots, x_n)\,{:}A_{\text{in}}(x_1)\ldots A_{\text{in}}(x_n){:}\,, \tag{10}$$

has been used (7,8).

Now we introduce interacting field operators:

$$A(x) = A_{\text{in}}(x) + \int \mathrm{d}^4y\,\Delta_R(x-y)\int_0^1 \mathrm{d}\lambda\,\exp[-i\lambda\eta]\vartheta(y)\exp[i\lambda\eta]\,. \tag{11}$$

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(5) Cf. H. LEHMANN, K. SYMANZIK, W. ZIMMERMANN: *Nuovo Cimento*, **1**, 425 (1955); in the following quoted as LSZ I.

(6) GREENBERG and WIGHTMAN (unpublished manuscript) have pointed out that the operator $A^f(t)$ may lead to states with infinite norm. To avoid this, they introduce time-averaged operators. This modification does not change the results derived in this paper. We wish to thank Drs. GREENBERG and WIGHTMAN for informing us of their results prior to publication.

(7) $\vartheta(y) = \delta\eta/\delta A_{\text{in}}(y)$ is the functional derivative of η with respect to $A_{\text{in}}(y)$, however, we shall not need this characterisation.

(8) The function $h_3(y;\, x_1, x_2)$ included in the sum is of course zero on the energy shell. It appears only in the extrapolation process.

It should be noted here that only

$$\int \Delta(x-y)\vartheta(y)\,d^4y \qquad \text{but not} \qquad \int \Delta_R(x-y)\vartheta(y)\,d^4y\,,$$

is uniquely determined if η is given. For it is clear from (10) that $\tilde{h}_{n+1}(k, k_1, \dots, k_n)$ has to be defined for $k^2 + m^2 \neq 0$ to determine $\vartheta(\eta)$. As has been discussed this extrapolation outside the energy shell can be carried out in different ways; we demand only that the conditions which $\tilde{h}_{n+1}(k, k_1, \dots, k_n)$ satisfies on the energy shell be fulfilled everywhere. Then it follows that $A(x)$ is invariant and hermitian. Moreover it satisfies the asymptotic condition (7) as is easily shown by using the Riemann-Lebesgue lemma on Fourier integrals.

We wish to stress that the described procedure of constructing relativistic field operators $A(x)$ corresponding to a given S-matrix does not involve any divergent steps. The necessary extrapolation of the functions $\tilde{h}_n(k_1, \dots, k_n)$ can of course be accomplished in a continuous and therefore finite manner.

Using the definition

$$(12) \qquad j(x) = -\int_0^1 d\lambda \exp[-i\lambda\eta]\vartheta(x)\exp[i\lambda\eta]\,,$$

we can write:

$$(13) \qquad \begin{cases} A_{\text{out}}(x) = A_{\text{in}}(x) + \int \Delta(x-y)j(y)\,dy\,, \\ A(x) \quad = A_{\text{in}}(x) - \int \Delta_R(x-y)j(y)\,dy\,, \\ (\Box - m^2)A(x) = j(x)\,. \end{cases}$$

Using the developed formalism we turn now to the concept of a causal scattering matrix. As is customary we call a field operator $A(x)$ causal — without discussing here the physical interpretation — if

$$(14) \qquad [A(x), A(y)] = 0\,. \qquad \text{for } (x-y)^2 > 0.$$

The operators $A(x)$ defined by (11) will of course not generally satisfy this condition. It is essential now that (14) is not only a condition on the extrapolation off the energy shell but also a condition for the scattering matrix, i.e. for the functions $\tilde{h}_n(k_1, \dots, k_n)$ for $k_1^2 = \dots = k_n^2 = -m^2$. If these functions have only the properties listed above, then it is in general impossible to extrapolate them continuously in such a manner that (14) is satisfied. This represents an additional requirement.

Hence we define a scattering matrix as causal if there exists at least one (continuous) extrapolation off the energy shell such that the resulting field $A(x)$ satisfies the commutator condition. It is an entirely open question whether any scattering matrix exists which is causal in the sense of this definition or whether this is too stringent a demand. Our notion of a causal S-matrix is equivalent to what is being used in investigations of local field theories. However, the above definition shows clearly that it must be possible in principle to express all consequences of the causality condition (14) for the energy shell in terms of S-matrix elements only.

We conclude this section by giving some formulae needed for the following. They are taken from LSZ I using a slightly different notation.

Let $\{f_\alpha(x)\}$ denote a complete and orthonormal system of positive frequency solutions to the Klein-Gordon equation.

$$(15)\qquad -i\int \mathrm{d}^3x f_\alpha(x)\, \overleftrightarrow{\frac{\partial}{\partial x_0}}\, f^*_\beta(x) = -i\int \mathrm{d}^3x \left\{ f_\alpha(x)\frac{\partial}{\partial x_0} f^*_\beta(x) - \frac{\partial}{\partial x_0} f_\alpha(x)\cdot f^*_\beta(x)\right\} = \delta_{\alpha\beta}\,,$$

$$(16)\qquad \sum_{\alpha=1}^{\infty} f_\alpha(x) f^*_\alpha(x') = i\Delta^+(x-x')\,.$$

The field operator $A(x)$ may be developed with respect to these functions

$$(17)\qquad A(x) = \sum_{\alpha=1}^{\infty}\{f_\alpha(x)A^\alpha(t) + f^*_\alpha(x)A^{\alpha *}(t)\}.$$

The coefficients are given by

$$(18)\qquad A^\alpha(t) = i\int \mathrm{d}^3x A(x)\, \overleftrightarrow{\frac{\partial}{\partial x_0}}\, f^*_\alpha(x)\,.$$

The corresponding operators $A^\alpha_{\mathrm{in}}(t)$ and $A^\alpha_{\mathrm{out}}(t)$ are of course time-independent:

$$A^\alpha_{\mathrm{in}}(t) = A^\alpha_{\mathrm{in}}\,, \qquad A^\alpha_{\mathrm{out}}(t) = A^\alpha_{\mathrm{out}}\,.$$

They constitute the limiting values of $A_\alpha(t)$ for $t \to \pm\infty$

$$(19)\qquad \lim_{t\to\pm\infty}\,(\Phi, A_\alpha(t)\Psi) = (\Phi, A^\alpha_{\substack{\mathrm{out}\\ \mathrm{in}}}\Psi)\,.$$

The operators A^α_{in}, A^α_{out} satisfy the commutation relations:

$$(20)\qquad \begin{cases} [A^\alpha_{\mathrm{in}}, A^{\beta *}_{\mathrm{in}}] = [A^\alpha_{\mathrm{out}}, A^{\beta *}_{\mathrm{out}}] = \delta_{\alpha\beta}\,, \\ [A^\alpha_{\mathrm{in}}, A^\beta_{\mathrm{in}}] = [A^\alpha_{\mathrm{out}}, A^\beta_{\mathrm{out}}] = 0\,. \end{cases}$$

$A_{\text{in}}^{\alpha *}$ creates an incoming particle with wave function $f_\alpha(x)$ and may be used to construct the complete orthonormal system

$$\Phi_{\text{in}}^{\alpha_1 \dots \alpha_k} = \frac{1}{\sqrt{p_{\alpha_1 \dots \alpha_k}}} A_{\text{in}}^{\alpha_1 *} \dots A_{\text{in}}^{\alpha_n *} \Omega \,, \tag{21}$$

$$(p_{\alpha_1 \dots \alpha_k} = n_1! \dots n_r! \text{ if } n_j \text{ of the indices } \alpha_i \text{ are equal}).$$

For an exact formulation of the asymptotic behavior it is essential to work with normalizable solutions of the Klein-Gordon equations (wave packets). For this reason we use a discrete orthonormal system in direct applications of the asymptotic condition. Final results can be expressed by means of plane waves without introducing errors. The development of $A_{\text{in}}(x)$ with respect to plane waves is given by (4). We define $a_{\text{in}}(k)$ by

$$a_{\text{in}}(k) = A_{\text{in}}(k) \,, \qquad a^*_{\text{in}}(k) = A_{\text{in}}(-k) \,. \qquad \text{for } k_0 > 0 \,. \tag{22}$$

It satisfies the invariant relations

$$[a_{\text{in}}(k), a^*_{\text{in}}(k')] = 2k_0 \, \delta(\boldsymbol{k} - \boldsymbol{k}') \,, \qquad [a_{\text{in}}(k), a_{\text{in}}(k')] = 0 \,, \tag{23}$$

$a_{\text{out}}(k)$ is defined in the same manner.

Finally, for the S-matrix we use the notation:

$$(\Phi_{\text{in}}^{p_1 \dots p_k}, S\Phi_{\text{in}}^{q_1 \dots q_l}) = (\Phi_{\text{out}}^{p_1 \dots p_k}, \Phi_{\text{in}}^{q_1 \dots q_l}) \,, \tag{24}$$

with

$$\Phi_{\substack{\text{out}\\ \text{in}}}^{p_1 \dots p_n} = a^*_{\substack{\text{out}\\ \text{in}}}(p_1) \dots a^*_{\substack{\text{out}\\ \text{in}}}(p_n) \Omega \,.$$

2. – Definition of retarded products.

We have defined an S-matrix to be causal if there exists an extrapolation such that the commutator condition (14) is satisfied. So far this is a formal definition that includes the S-matrices of the conventional local field theories.

However, we do not know if this concept of causality is the one realized in nature. Therefore it is necessary to confront the properties of (in the above sense) causal S-matrices with experiment. The commutator condition (14) does not immediately yield explicit properties of the scattering matrix; non-trivial mathematical steps must be taken to derive practically useful consequences from it. We shall discuss some formal concepts which are useful — and partly have been used, especially by POLKINGHORNE (⁹) — for an evaluation of the commutator condition.

In this section we define and investigate a retarded product of field operators. For causality problems this product seems more promising than e. g. the time-ordered product. We define the retarded product of *two* field operators as the retarded commutator:

$$R(x;\, y) = -\, i\theta(x-y)[A(x),\, A(y)]\,. \tag{26}$$

This, expression is familiar from the theory of dispersion relations. The retarded product of $n+1$ field operators is defined by

$$\left\{\begin{aligned} &n=0: && R(x) = A(x)\,, \\ &n\geqslant 1: && R(x;\, x_1 \dots x_n) = \\ & && = (-i)^n \sum \theta(x-x_1)\dots\theta(x_{n-1}-x_n)[\dots[A(x),\, A(x_1)]\dots A(x_n)]\,, \end{aligned}\right. \tag{27}$$

which will turn out to be a natural generalization of (26). The summation is taken over all permutations of the n coordinates x_i.

We call the vacuumexpectation values

$$r(x;\, x_1 \dots x_n) = (\Omega,\, R(x;\, x_1 \dots x_n)\Omega)\,, \tag{28}$$

retarded functions (r-functions) of the field operators. Three simple properties of R-products follow from the definition:

Retardation: $R(x;\, x_1,\, \dots\, x_n)$ vanishes if one or several of the times x_{i0} is larger than x_0.

Symmetry: $R(x;\, x_1,\, \dots\, x_n)$ is symmetric in the co-ordinates x_i.

Hermiticity: The R-product of hermitian operators is hermitian; i.e. the retarded functions are real.

The R-products can be generated from the functional (¹⁰)

$$\mathcal{R}\{x;\, J\} = -\, i\mathcal{T}^*\{J\}\, \frac{\delta}{\delta J(x)}\, \mathcal{T}\{J\}\,; \qquad \mathcal{T}\{J\} = T\exp\left[\left(i\int \mathrm{d}x J(x)A(x)\right)\right], \tag{29}$$

(⁹) POLKINGHORNE (*Nuovo Cimento*, **4**, 216 (1956) and *Proc. Cambr. Phil. Soc.*, **53**, 260 (1957)) has introduced a general type of product which depends not only on the space-time coordinates of the field operators, but also on additional parameters. For special choice of these parameters Polkinghorne's product is identical with our R-product, as defined below in eq. (27).

(¹⁰) Cf. LSZ I, appendix.

by means of functional differentiation with respect to $J(x)$.

$$R(x; x_1 \dots x_n) = \left.\frac{\delta^n \mathcal{R}\{x, \mathring{J}\}}{\delta J(x_1) \dots \delta J(x_n)}\right|_{J=0} . \tag{30}$$

We state some more properties of R-products which will then be used to discuss the significance of R-products for the derivation of causality conditions.

1) Invariance properties.

$$\begin{cases} R(x + a; x_1 + a, \dots, x_n + a) = \exp[-iPa]R(x; x_1 \dots x_n) \exp[iPa] , \\ \qquad\qquad \text{if} \quad A(x + a) = \exp[-iPa]A(x) \exp[iPa] . \end{cases} \tag{31}$$

The retarded functions depend therefore only on the coordinate differences:

$$r(x; x_1 \dots x_n) = r(x - x_1, \dots, x - x_n) . \tag{32}$$

The retarded product of Lorentz invariant and causal field operators is moreover invariant under proper Lorentz transformations L:

$$\begin{cases} R(Lx; Lx_1, \dots, Lx_n) = U(L)R(x; x_1 \dots x_n)U(L)^{-1} , \\ r(Lx; Lx_1, \dots, Lx_n) = r(x; x_1 \dots x_n) , \end{cases} \tag{33}$$

if $A(Lx) = U()A(x)U(L)^{-1}$ and $[A(x)A(y)] = 0$ for $(x - y)^2 > 0$. (33) is proved using (29) or applying Jacobi identities to (27).

2) The asymptotic condition at $t = -\infty$ gives rise to the following commutation relations of R-products with the incoming field operator: $(K_z \equiv \square_z - m^2)$

$$[R(x; x_1 \dots x_n), A_{\text{in}}(z)] = -i \int \mathrm{d}z' \Delta(z - z')K_{z'} R(x; x_1 \dots x_n, z'), \tag{34}$$

or

$$[R(x; x_1 \dots x_n), A^*_{\text{in}}(k)] = \frac{\varepsilon(k)}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}z \exp[ikz]K_z R(x; x_1 \dots x_n, z) . \tag{34'}$$

To give a rigorous proof of these relations we use a discrete orthonormal system $\{f_\alpha(x)\}$ (cfr. (17)). Then (3) takes the form

$$[R(x; x_1 \dots x_n), A^{\alpha *}_{\text{in}}] = \int \mathrm{d}z\, f_\alpha(x) K_z R(x; x_1 \dots x_n, z) \;\; (^{11,12}) . \tag{34''}$$

---

(11) K. Nishijima (to be published) has suggested a formulation of the asymptotic condition which avoids the operators of incoming and outgoing fields and may be ob-

Now we note that as a consequence of the asymptotic condition at $t = -\infty$ we have for arbitrary normalizable states:

$$\begin{aligned}(\Phi, [R(x; x_1 \dots x_n), A_{\text{in}}^{\alpha *}]\Psi) &= \lim_{z_0 \to -\infty} (\Phi, [R(x; x_1 \dots x_n), A^{\alpha *}(z_0)]\Psi) = \\ &= \lim_{z_0 \to -\infty} i \int \mathrm{d}^3 z (\Phi, [R(x; x_1 \dots x_n) A(z)]\Psi) \frac{\overleftrightarrow{\partial}}{\partial z_0} f_\alpha(z) = \\ &= - \lim_{z_0 \to -\infty} \int \mathrm{d}^3 z (\Phi, R(x; x_1 \dots x_n, z)\Psi) \frac{\overleftrightarrow{\partial}}{\partial z_0} f_\alpha(z) = \\ &= \int \mathrm{d}z \frac{\partial}{\partial z_0} \left\{ (\Phi, R(x; x_1 \dots x_n, z)\Psi) \frac{\overleftrightarrow{\partial}}{\partial z_0} f_\alpha(z) \right\} = \\ &= \int \mathrm{d}z\, f_\alpha(z) K_z (\Phi, R(x; x_1 \dots x_n, z)\Psi),\end{aligned}$$

since the boundary term

$$\lim_{z_0 \to +\infty} \int \mathrm{d}^3 z (\Phi, R(x; x_1 \dots x_n z)\Psi) \frac{\overleftrightarrow{\partial}}{\partial z_0} f_\alpha(z) ,$$

vanishes due to the retarded character of the R-product.

Iteration of (34′) gives the result:

$$\begin{aligned}(35) \qquad & \varepsilon(k_1) \dots \varepsilon(k_n) [\dots [R(x; x_1 \dots x_m), A_{\text{in}}^*(k_1)] \dots A_{\text{in}}^*(k_n)] = \\ & = \frac{1}{(2\pi)^{\frac{3}{2}n}} \int \mathrm{d}z_1 \dots \mathrm{d}z_n \exp [i \sum k_i z_i] K_{z_1} \dots K_{z_n} R(x; x_1 \dots x_m\, z_1 \dots z_n).\end{aligned}$$

3) Using the asymptotic condition both at $t = -\infty$ and $t = +\infty$ we obtain commutation relations of the outgoing field with the incoming field (¹³):

$$(36) \qquad [A_{\text{out}}(x) A_{\text{in}}(y)] = i\Delta(x - y) - i \int \mathrm{d}x' \, \mathrm{d}y' \, \Delta(x - x') \Delta(y - y') K_{x'} K_{y'} R(x'; y') .$$

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tained by taking matrix elements of (34″) with respect to $\Phi_{\text{in}}^{\alpha_1 \dots \alpha_l}$, $\Phi_{\text{in}}^{\beta_1 \dots \beta_l}$. This modified procedure is useful to include the possibility of bound states in the formalism of r-functions, especially in connection with the completeness of the incoming and outgoing states (compare Appendix). We wish to thank Dr. NISHIJIMA for informing us of his results prior to publication.

(¹²) A special case of this formula is the commutation relation between the current and the incoming field operator which was derived in the same way by M. CINI and S. FUBINI: *Nuovo Cimento*, **2**, 860 (1955).

(¹³) Compare also S. SCHWEBER: *Nuovo Cimento*, **2**, 397 (1955).

This relation follows by applying (13) and (34):

$$\begin{aligned}[][A_{\text{out}}(x)A_{\text{in}}(y)] - [A_{\text{in}}(x)A_{\text{in}}(y)] &= \int \mathrm{d}x'\,\Delta(x-x')K_{x'}[A(x')A_{\text{in}}(y)] = \\ &= -i\int \mathrm{d}x'\,\Delta(x-x')K_{x'}\int \mathrm{d}y'\,\Delta(y-y')K_{y'}R(x';y') = \\ &= -i\int \mathrm{d}x'\,\mathrm{d}y'\,\Delta(x-x')\Delta(y-y')K_{x}K_{y'}R(x';y') \quad ({}^{14})\end{aligned}$$

By Fourier transformation (36) can also be written in the equivalent form

$$(36')\qquad \begin{cases} [a_{\text{out}}(k)a^{*}_{\text{in}}(k')] = 2k_0\,\delta(\boldsymbol{k}-\boldsymbol{k}') - \\ \qquad\qquad -\dfrac{i}{(2\pi)^3}\displaystyle\int \mathrm{d}x\,\mathrm{d}x'\,\exp\,[i(k'x'-kx)]K_xK_{x'}R(x;x')\,, \\ \\ [a_{\text{out}}(k)a_{\text{in}}(k')] = \dfrac{i}{(2\pi)^3}\displaystyle\int \mathrm{d}x\,\mathrm{d}x'\,\exp\,[-i(kx+k'x')]K_xK_{x'}R(x;x')\,. \end{cases}$$

The more general formula

$$(37)\qquad \begin{aligned}&[\ldots[a_{\text{out}}(k)-a_{\text{in}}(k),\,a^{*}_{\text{in}}(k_1)]\ldots a^{*}_{\text{in}}(k_n)] = \\ &= -\frac{i}{(2\pi)^{\frac{3}{2}(n+1)}}\int \mathrm{d}x\,\mathrm{d}x_1\ldots \mathrm{d}x_n\,\exp\,[i(\textstyle\sum k_ix_i-kx)]K_{x_1}\ldots K_{x_n}R(x;x_1\ldots x_n)\,,\end{aligned}$$

follows in the same way from (13) and (35).

3. – Simple applications of R-products.

As an example for the application of R-products we consider first the S-matrix element for the scattering of two bosons.

$$S(k',p';k,p) = (\Phi^{k'p'}_{\text{out}},\,\Phi^{kp}_{\text{in}})\,.$$

The relation ($S = 1 - iT$)

$$(38)\qquad T(k',p';k,p) = \frac{1}{(2\pi)^3}\int \mathrm{d}x\,\mathrm{d}x'\,\exp\,[i(kx-k'x')]K_xK_{x'}(\Phi^{p'}_{\text{in}},\,R(x';x)\Phi^{p}_{\text{in}})\,,$$

---

(14) The interchange of the order of integration in this double integral can be justified with the aid of the commutators of the incoming and outgoing field. The following theorem can be proved: The order of integration in the above integral may be interchanged if and only if the incoming and outgoing fields satisfy the same commutation relations. The same applies to the order of integration in (35) and (37).

which forms the basis for derivations of Goldberger's dispersion (¹⁵) relations follows quite simply if we only assume that the field operator fulfills the asymptotic condition.

Since

$$\Phi^{k}_{\text{out}} = \Phi^{k}_{\text{in}} = \Phi^{k}\,,$$

$$\begin{aligned}(\Phi^{k'p'}_{\text{out}}, \Phi^{kp}_{\text{in}}) &= (\Phi^{p'}_{\text{out}}, a_{\text{out}}(k')a^{*}_{\text{in}}(k)\Phi^{p}_{\text{in}}) = (\Phi^{p'}, a_{\text{out}}(k')a^{*}_{\text{in}}(k)\Phi^{p})\,,\\ &= (\Phi^{p'}, [a_{\text{out}}(k'), a^{*}_{\text{in}}(k)]\Phi^{p}) + 4k_0p_0\,\delta(\boldsymbol{k}'-\boldsymbol{p})\,\delta(\boldsymbol{p}'-\boldsymbol{k})\,.\end{aligned}$$

Insertion of equation (36′) for the commutator between outgoing and incoming fields

$$\begin{aligned}(\Phi^{p'}, [a_{\text{out}}(k'), a^{*}_{\text{in}}(k)]\Phi^{p}) &= 4k_0p_0\,\delta(\boldsymbol{k}'-\boldsymbol{k})\,\delta(\boldsymbol{p}'-\boldsymbol{p}) -\\ &\quad - \frac{i}{(2\pi)^3}\int \mathrm{d}x\,\mathrm{d}x'\,\exp[i(kx-k'x')]K_xK_{x'}(\Phi^{p'}, R(x';x)\Phi^{p})\,,\end{aligned}$$

yields the relation (38).

This expresses the scattering amplitude directly as a matrix element of $R(x';x)$ between one-particle states. If the S-matrix is causal, i. e. $A(x)$ satisfies the commutator condition, $(\Phi^{p'}, R(x';x)\Phi^{p})$ is an invariant function that vanishes outside the forward cone of $x'-x$. This property, together with some consequences of the energy momentum spectrum, suffices in the case of forward scattering ($p'=p$, $k'=k$) to prove the dispersion relation for the forward amplitude and thereby obtain an explicit property of a causal S-matrix.

For non-forward scattering dispersion relations do not follow directly from (38). To obtain in this case enough consequences of causality for the scattering amplitude one has to reduce it to the vacuum expectation value of $R(x';x,y,y')$. This can be done as follows:

$$\begin{aligned}(\Phi^{p'}, R(x';x)\Phi^{p}) &= (\Phi^{p'}, R(x';x)a^{*}_{\text{in}}(p)\Omega) =\\ &= 2p_0\,\delta(\boldsymbol{p}-\boldsymbol{p}')(\Omega, R(x';x)\Omega) + (\Phi^{p'}, [R(x';x)a^{*}_{\text{in}}(p)]\Omega)\,,\\ &= 2p_0\,\delta(\boldsymbol{p}-\boldsymbol{p}')(\Omega, R(x';x)\Omega) + \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}y\,\exp[ipy]K_y(\Phi^{p'}, R(x';xy)\Omega)\,.\end{aligned}$$

Again we have used the commutator (34′) of the R-product with the incoming

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(¹⁵) M. L. Goldberger: *Phys. Rev.*, **99**, 979 (1955); R. Oehme: *Nuovo Cimento*, **4**, 1316 (1956); K. Symanzik: *Phys. Rev.*, **105**, 743 (1957).

field. In the same manner

$$(\Phi^{p'}, R(x' : xy)\Omega) = (\Omega, [a_{\text{in}}(p'), R(x'; x, y)]\Omega) = \\ = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}y' \exp[-ip'y'] K_{y'}(\Omega, R(x'; x, y, y')\Omega) .$$

Combining these equations we obtain the desired relation

$$(39) \qquad T(k', p'; kp) = \\ = \frac{1}{(2\pi)^6} \int \mathrm{d}x\, \mathrm{d}x'\, \mathrm{d}y\, \mathrm{d}y' \exp[i(kx + py - k'x' - p'y')] K_x K_x K_y K_{y'} r(x'; x, y, y') .$$

According to Bogoljubow (16) dispersion relations for non-forward scattering can be derived from this equation if in addition to the commutator condition certain consequences of the energy-momentum spectrum are taken into account, which follow from the identity

$$(40) \qquad R(x'; xyy') = i\theta(x' - x)\{[R(x', yy'), A(x)] + [R(x'; y)R(x; y')] + \\ + [R(x'; y')R(x; y)] + [A(x'), R(x; yy')]\} .$$

Turning to more general processes we wish to point out that all S-matrix elements with two incoming particles (this is not a serious restriction in view of experimental possibilities) and an arbitrary number of outgoing particles may be expressed by means of retarded products. Assuming invariance under space-time inversion, we have for $n \geqslant 3$ the relation

$$(41) \qquad S(p_1, \dots, p_n; q_1, q_2) = S(q_1, q_2; p_1 \dots p_n) = (\Phi^{q_1 q_2}_{\text{out}}, \Phi^{p_1 \dots p_n}_{\text{in}}) =$$

$$(42) \qquad = \frac{-i}{(2\pi)^{\frac{3}{2}n}} \int \mathrm{d}x_2 \dots \mathrm{d}x_n\, \mathrm{d}y_2 \exp[i(p_2 x_2 + \dots + p_n x_n) - iq_2 y_2] \cdot \\ \cdot K_{x_2} \dots K_{x_n} K_{y_2} (\Phi^{q_1}, R(y_2; x_2 \dots x_n)\Phi^{p_1}) =$$

$$(43) \qquad = \frac{-i}{(2\pi)^{\frac{3}{2}(n+2)}} \int \mathrm{d}x_1 \dots \mathrm{d}x_n\, \mathrm{d}y_1\, \mathrm{d}y_2 \exp[i(\sum_1^n p_i x_i - \sum_1^2 q_i y_i)] \cdot \\ \cdot K_{x_1} \dots K_{x_n} K_{y_1} K_{y_2} r(y_2; y_1 x_1 \dots x_n) =$$

To derive (42), we note that an elementary calculation yields

$$(\Phi^{q_1 q_2}_{\text{out}}, \Phi^{p_1 \dots p_n}_{\text{in}}) = [\Phi^{q_1}, (\dots [a_{\text{out}}(q_2) - a_{\text{in}}(q_2), a^*_{\text{in}}(p_2)] \dots a^*_{\text{in}}(p_2))\Phi^{p_1}] .$$

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(16) N. N. Bogoljubow: *Report at the Int. Conference on Theor. Physics, Seattle*, 1956.

(17) K. Symanzik: *Phys. Rev.*, **105**, 743 (1957).

In this formula we have only to insert (37) on the right-hand side to obtain (42). Eq. (43) follows easily if we apply the commutation relation (34′) twice to (42).

The equations treated in this section are examples of reduction formulae for retarded products. Matrix elements of R-products (and the S-matrix) are reduced to other matrix elements of R-products between states with a smaller number of particles; in particular to vacuum expectation values.

Another example is the relation

$$(44)\qquad (\Phi^p, A(x)\Phi^{p'}) = (\Omega, [a_{\text{in}}(p), A(x)]\Phi^{p'}) = \\ = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}y \exp[-ipy]K_y(\Omega, R(x; y)\Phi_{p'}) ,$$

which has been used to discuss the analytic properties of the vertex function (¹⁴).

Very general reduction formulae follow if matrix elements between arbitrary incoming states are taken in eq. (35) and (37). We do not consider them here in detail; they are closely connected with expansions of $A(x)$ and $R(x; x_1 \dots x_n)$ with respect to incoming fields which we shall discuss in a separate paper.

APPENDIX

In this paper we have disregarded the possible existence of stable bound states. However, it may be shown that the results derived in Sect. 2 and 3 are generally valid. The asymptotic condition

$$(7)\qquad \lim_{t\to\pm\infty}(\Phi, A^f(t)\Psi) = (\Phi, A^f_{\substack{\text{out}\\ \text{in}}}(t)\Psi) ,$$

remains unchanged even in theories with bound states (¹⁸). Only, in this case the incoming and outgoing operators do not give an irreducible representation (¹⁹) of the free field commutation relations (3); i.e. the orthogonal system (21) obtained by applying the operators A_{in}, A_{out} to the vacuum state is not a complete system. Instead we require merely that the field operators $A(x)$ form a complete operator ring which means that all states can be constructed by multiple application of $A(x)$ to the vacuum vector and superposition of the resulting states.

(⁸¹) Compare K. SYMANZIK: *Phys. Rev.*, **105**, 743 (1957).

(¹⁹) In LSZ I we postulated the irreducibility of A_{in}, A_{out} in connection with the asymptotic condition; compare footnote (⁵) of that paper.

In Sect. 2 we used only the limit relation (7) and the commutator (3) of the free fields but not the completeness of the system (21). Therefore we may take over the commutation relations (34) and (36) and the applications made in Sect. 3 without change.

If we study in particular the interaction of a nucleon field $\psi(x)$ with a meson field $A_\alpha(x)$, taking into account existing bound states, the well-known expression for the meson-nucleon scattering amplitude

$$T(k', \alpha', p'; k, \alpha, p) = \frac{-i}{(2\pi)^3}\int \mathrm{d}x\,\mathrm{d}x' \exp[i(kx - k'x')] \cdot \\ \cdot K_x K_{x'} \theta(x' - x)(\Phi_{\text{in}}^{p'}, [A_{\alpha'}(x')A_\alpha(x)]\Phi_{\text{in}}^{p}) , \tag{45}$$

follows in the same manner as eq. (38).

Quantum Field Theory in Terms of Vacuum Expectation Values

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Vacuum expectation values of products of neutral scalar field operators are discussed. The properties of these distributions arising from Lorentz invariance, the absence of negative energy states and the positive definiteness of the scalar product are determined. The vacuum expectation values are shown to be boundary values of analytic functions. Local commutativity of the field is shown to be equivalent to a symmetry property of the analytic functions. The problem of determining a theory of a neutral scalar field given its vacuum expectation values is posed and solved.

1. INTRODUCTION

RECENT work in relativistic quantum field theory has made heavy use of certain basic singular functions defined as vacuum expectation values of products of fields taken at various space-time points. In this paper, we present some results of a systematic study of relativistic field theory based on such vacuum expectation values. For simplicity, we treat the case of a neutral scalar field interacting with itself. The methods used have generalizations in any field theory.

The objects of our attentions are the singular functions:

$$F^{(n)}(x_1,\cdots x_n)=(\Psi_0,\phi(x_1)\phi(x_2)\cdots\phi(x_n)\Psi_0),$$

where Ψ_0 is the vacuum state, assumed to be the unique state of energy and momentum zero, and $\phi(x)$ is a neutral scalar field. As is well known, $F^{(n)}$ has to be understood as a distribution in the sense of L. Schwartz. It is a linear functional which gives a complex number for each infinitely differentiable function $f(x_1,\cdots x_n)$ which vanishes outside a bounded region of space time:

$$F^{(n)}(f)=\int d^4x_1\cdots d^4x_n f(x_1,\cdots x_n)F^{(n)}(x_1,\cdots x_n).$$

(We call such f testing functions.) Furthermore, $F^{(n)}(f_k)\to 0$ if a sequence of testing functions $f_k(x_1,\cdots x_n)$ (vanishing outside a fixed bounded region) and all their derivatives converge to zero uniformly in space time. We shall study the structure of $F^{(n)}$, exploiting systematically the Lorentz transformation properties of $\phi(x)$ which are given by

$$U(a,\Lambda)\phi(x)U(a,\Lambda)^{-1}=\phi(\Lambda x+a). \qquad (1)$$

Here, $\{a,\Lambda\}$ is an element of the inhomogeneous Lorentz group meaning the operation of transforming by the homogeneous Lorentz transformation, Λ, followed by translating by a. $U(a,\Lambda)$ is the corresponding unitary or antiunitary operator which yields the transformed wave functions. We shall also determine the consequences for $F^{(n)}$ of the assumptions that no negative energy states exist in the theory and that $\phi(x)$ is a local field. Finally, we shall show how, given a set $F^{(n)}$, $n=1, 2, \cdots$, one can construct a theory of a neutral scalar field which has these $F^{(n)}$ as its vacuum expectation values. We do not show that any set of $F^{(n)}$ outside of those determined by the free field actually exists.

2. CONSEQUENCES OF LORENTZ INVARIANCE

For Lorentz transformations without time inversion, we know that $U(a,\Lambda)$ is unitary. Thus,

$$\begin{aligned}F^{(n)}(x_1,\cdots x_n)&=(U(a,\Lambda)\Psi_0,U(a,\Lambda)\phi(x_1)\cdots\phi(x_n)\Psi_0)\\&=(\Psi_0,\ \phi(\Lambda x_1+a)\cdots\phi(\Lambda x_n+a)\Psi_0),\end{aligned}$$

and therefore

$$F^{(n)}(x_1,\cdots x_n)=F^{(n)}(\Lambda x_1+a,\ \cdots\Lambda x_n+a) \qquad (2)$$

for $\{a,\Lambda\}$ without time inversion. Here we have used (1) and the Lorentz invariance of the vacuum state:

$$U(a,\Lambda)\Psi_0=\Psi_0. \qquad (3)$$

For Lorentz transformations with time inversion, we have, on the other hand,[1]

$$\begin{aligned}F^{(n)}(x_1,\cdots x_n)&=(U(a,\Lambda)^{-1}\Psi_0,\phi(x_1)\cdots\phi(x_n)\Psi_0)\\&=[(\Psi_0,U(a,\Lambda)\phi(x_1)\cdots\phi(x_n)\Psi_0)]^*\\&=[(\Psi_0,\phi(\Lambda x_1+a)\cdots\phi(\Lambda x_n+a)\Psi_0)]^*.\end{aligned}$$

Thus,

$$F^{(n)}(x_1,\cdots x_n)=[F^{(n)}(\Lambda x_1+a,\cdots\Lambda x_n+a)]^* \qquad (4)$$

for $\{a,\Lambda\}$ with time inversion.

A somewhat similar relation is derived from the hermiticity of $\phi(x)$:

$$\begin{aligned}F^{(n)}(x_1,\cdots x_n)&=[(\Psi_0,(\phi(x_1)\cdots\phi(x_n))^*\Psi_0)]^*\\&=[(\Psi_0,\phi(x_n)\cdots\phi(x_1)\Psi_0)]^*.\end{aligned}$$

Thus,

$$F^{(n)}(x_1,\cdots x_n)=[F^{(n)}(x_n,\cdots x_1)]^*. \qquad (5)$$

3. CONSEQUENCES OF THE ABSENCE OF NEGATIVE ENERGY STATES

From (2), we see that $F^{(n)}$ is a function only of the differences of the $x_1,\cdots x_n$. We shall therefore write

$$F^{(n)}=F^{(n)}(\xi_1,\cdots\xi_{n-1}),$$

---

[1] For convenience in printing we shall often denote the complex conjugate, $\bar{a}$, of a number a, by $[a]^*$; this latter notation is the same as the notation we use to denote adjoint of an operator.

where

$$\xi_1=x_1-x_2,\quad \xi_2=x_2-x_3,\ \cdots\quad \xi_{n-1}=x_{n-1}-x_n.$$

We shall assume that $F^{(n)}$ has a Fourier transform and shall show that if

$$F^{(n)}(\xi_1,\cdots\xi_{n-1})=\int\exp\left(-i\sum_{j=1}^{n-1}p_j\cdot\xi_j\right)$$
$$\times G^{(n)}(p_1,\cdots p_{n-1})d^4p_1\cdots d^4p_{n-1},$$

then $G^{(n)}$ vanishes unless the p_j satisfy

$$p_j^2\equiv(p_j^0)^2-\mathbf{p}_j^2\geq 0,\quad p_j^0\geq 0,$$

by virtue of the assumption that no negative-energy states exist.

If Ψ is an arbitrary state, its component of momentum p is

$$\int e^{-ip\cdot a}d^4aU(a,1)\Psi.$$

Thus, by our hypothesis that no negative energy states exist,

$$\int e^{ip\cdot a}d^4aF^{(n)}(\xi_1,\cdots\xi_{j-1},\,\xi_j+a,\,\xi_{j+1}\cdots\xi_{n-1})$$
$$=\int e^{ip\cdot a}d^4a(\Psi_0,\,\phi(x_1)\cdots\phi(x_j)$$
$$\times\phi(x_{j+1}-a)\cdots\phi(x_n-a)\Psi_0)$$
$$=\Bigg(\Psi_0,\,\phi(x_1)\cdots\phi(x_j)\int e^{ip\cdot a}d^4aU(-a,\,1)$$
$$\times\phi(x_{j+1})\cdots\phi(x_n)\Psi_0\Bigg)$$

vanishes unless p is within or on the forward light cone. Therefore, $G^{(n)}(p_1,\cdots p_{n-1})$ vanishes if any of its arguments lie outside the forward light cone. In the special case $n=2$, this result is well known.

It is an important consequence of this property of the $G^{(n)}$ that the distributions $F^{(n)}$ are boundary values of analytic functions. This result is displayed in and simultaneously proved by the formula

$$F^{(n)}(\xi_1-i\eta_1,\cdots\xi_{n-1}-i\eta_{n-1})$$
$$=\int\exp\left(-i\sum_{j=1}^{n-1}p_j\cdot(\xi_j-i\eta_j)\right)$$
$$\times G^{(n)}(p_1,\cdots p_{n-1})d^4p_1\cdots d^4p_{n-1},$$

where the four-vectors η_j are restricted to lie in the future light cone. The $8(n-1)$-dimensional open region thus defined in the $8(n-1)$-dimensional space of the components of the $\xi_1,\cdots\xi_{n-1},\eta_1,\cdots\eta_{n-1}$ is called the future tube. Thus, $F^{(n)}(\xi_1,\cdots\xi_{n-1})$ is a boundary value of a function analytic in the future tube.

We introduce the notation

$$z_{j,\,j+1}=\xi_j-i\eta_j.$$

Equation (2) implies the Lorentz invariance of $G^{(n)}$:

$$G^{(n)}(p_1,\cdots p_{n-1})=G^{(n)}(\Lambda p_1,\cdots\Lambda p_{n-1}),$$

for Λ without time inversion, and Eq. (4):

$$G^{(n)}(p_1,\cdots p_{n-1})=[G^{(n)}(-\Lambda p_1,\cdots-\Lambda p_{n-1})]^*,$$

for Λ with time inversion. Consequently, we have throughout the tube

$$F^{(n)}(z_{12},\cdots z_{n-1,\,n})=F^{(n)}(\Lambda z_{12},\cdots\Lambda z_{n-1,\,n})\tag{6}$$

for Λ without time inversion and

$$F^{(n)}(z_{12},\cdots z_{n-1,\,n})=[F^{(n)}(\Lambda\bar{z}_{12},\cdots\Lambda\bar{z}_{n-1,\,n})]^*\tag{7}$$

for Λ with time inversion.

We use the following theorem, whose proof will be published in another paper.

Theorem.—A function $f(z_1,\cdots z_n)$ of n four-vector variables $z_1,\cdots z_n$ analytic in the tube:

$$-\infty<\mathrm{Re}\,z_{i\mu}<\infty,\ \mathrm{Im}\,z_i\text{ in the future cone,}$$

and invariant under the homogeneous Lorentz group without time inversion:

$$f(z_1,\cdots z_n)=f(\Lambda z_1,\cdots\Lambda z_n)$$

is a function of the scalar products $z_j\cdot z_k$, j, $k=1,\,2,\,\cdots n$. It is analytic in the complex manifold over which the scalar products vary when the vectors $z_1,\ \cdots z_n$ vary over the future tube.

If we introduce the complex vectors

$$z_{ij}=\sum_{k=i}^{j-1}z_{k,\,k+1}\quad\text{for}\quad i<j,$$

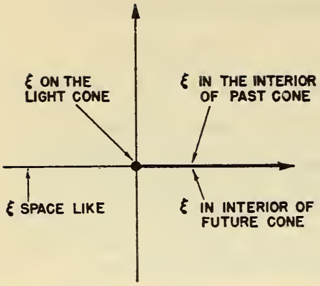
then the theorem tells us that the $F^{(n)}$ are analytic functions of the squares of the lengths of these vectors

$$F^{(n)}=F^{(n)}(z_{ij}^2).\tag{8}$$

As the variables $z_{j,\,j+1}$ vary over the tube each variable z_{ij}^2 varies in an open set of the complex plane. From the explicit form

$$z^2=(\xi-i\eta)^2=\xi^2-\eta^2-2i\xi\cdot\eta$$

one can easily see that this set fills the entire plane except for the positive real axis and the origin. The situation is indicated in detail in Fig. 1. Of course, the z_{ij}^2 are not independent, so that the manifold they define is not just the topological product of cut planes. The theory of a scalar field can be regarded as a boundary value problem for this set of analytic functions. We give an example of such a set, those arising from the theory of a scalar field $\phi(x)$ satisfying $(\Box+m^2)\times\phi(x)=0$ and the commutation rules $[\phi(x),\phi(x')]_-$



ξ ON THE LIGHT CONE
ξ IN THE INTERIOR OF PAST CONE
ξ SPACE LIKE
ξ IN INTERIOR OF FUTURE CONE

FIG. 1. Domain of variation of the complex variable $z^2 \equiv (\xi - i\eta)^2 = \xi^2 - \eta^2 - 2i\xi\cdot\eta$, when ξ varies over all space time and η in the future cone. The labeled points indicate values of z^2 in the physical limit $\eta \to 0$ for typical positions of ξ.

$= i^{-1}\Delta(x-x')$:

$$F^{(n)} = 0, \qquad n \text{ odd}$$

$$F^{(n)}(z_{ij}^2) = \left(\frac{m^2}{8\pi i}\right)^{n/2} \sum \frac{H_1^{(1)}(m(z_{12}^2)^{\frac12})}{m(z_{12}^2)^{\frac12}} \times \frac{H_1^{(1)}(m(z_{34}^2)^{\frac12})}{m(z_{34}^2)^{\frac12}} \cdots \frac{H_1^{(1)}(m(z_{n-1,n}^2)^{\frac12})}{m(z_{n-1,n}^2)^{\frac12}}, \quad n \text{ even.} \tag{9}$$

The sum in (9) goes over all "pairings" of the sequence $(1\cdots n)$, a "pairing" being a division of the set into $(n/2)$ disjoint subsets. The formulas assume a more familiar aspect if one goes over to the physical limit in which

$$\left(\frac{m^2}{8\pi}\right) \frac{H_1^{(1)}(m(z^2)^{\frac12})}{m(z^2)^{\frac12}} \to \Delta^{(+)}(\xi).$$

4. CONSEQUENCES OF THE COMMUTATION RULES

The local commutation rules

$$[\phi(x), \phi(y)]_- = 0, \quad (x-y)^2 < 0,$$

imply that

$$F^{(n)}(x_1, \cdots x_j, x_{j+1}, \cdots x_n) = F^{(n)}(x_1, \cdots x_{j+1}, x_j \cdots x_n) \tag{10}$$

as long as x_j and x_{j+1} are space-like separated points. These relations can be extended by analytic continuation to relations of the analytic functions $F^{(n)}(z_{ij}^2)$:

$$F^{(n)}(z_{ik}^2) = F^{(n)}(Pz_{ik}^2), \tag{11}$$

where P stands for the operation of permuting the subscripts j and $j+1$, and, by definition, $z_{ik}^2 = z_{ki}^2$. The proof is simple. (11) coincides with (10) for all z_{ik}^2 on the negative real axis. Consequently, it holds everywhere by analytic continuation. Thus, the local property of a field $\phi(x)$ is characterized by a global symmetry relation on the analytic functions $F^{(n)}$.

Unlike the local commutation rules, the more special canonical commutation rules require a certain amount of heuristic juggling before a translation in terms of of analytic functions is possible.

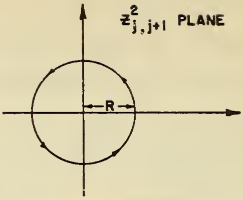


FIG. 2. Contour for the integration of Eq. (14).

We start from the equation

$$\left[\frac{\partial\phi}{\partial(ct)}(t,\mathbf{x}_1), \phi(t,\mathbf{x}_2)\right]_- = \frac{1}{i}\delta(\mathbf{x}_1 - \mathbf{x}_2). \tag{12}$$

Integrating over all space, we have

$$\int d^3\mathbf{x}_1 \left[\frac{\partial\phi}{\partial(ct)}(t,\mathbf{x}_1), \phi(t,\mathbf{x}_2)\right]_- = i^{-1}$$

or

$$\lim_{\Delta t\to 0} (2c\Delta t)^{-1} \left\{ \int d^3\mathbf{x}_1 [\phi(t+\Delta t, \mathbf{x}_1), \phi(t,\mathbf{x}_2)]_- - \int d^3\mathbf{x}_1 [\phi(t-\Delta t, \mathbf{x}_1), \phi(t,\mathbf{x}_2)]_- \right\} = i^{-1}.. \tag{13}$$

Now we assume that, in the limit, the only contribution to this integral will come from the singularity on the light cone, an assumption whose validity will be discussed at the end of this section. For the first term in the first commutator of (13), the integral then can be written

$$\int d^3x_1 \cdots = 4\pi \int_{(c\Delta t)^2}^{0} d\xi^2 \{-\tfrac{1}{2}[(c\Delta t)^2 - \xi^2]^{\frac12}\} \cdots,$$

where $\xi = (c\Delta t, \mathbf{x}_1 - \mathbf{x}_2)$. In the complex plane of Fig. 1, this corresponds to an integration just below the real axis from $(c\Delta t)^2$ to the origin. From the first term in the second commutator we get a similar integral just above the real axis. This suggests that the two terms contribute a contour integral around the branch cut. The second terms combine to given an equal contribution. Thus, it seems plausible that the canonical commutation relations are equivalent to the requirement that

$$\lim_{R\to 0} \int_{C(R)} dz_{j,j+1}^2 F^{(n)}(z_{kl}^2) = \frac{1}{2\pi i} F^{(n-2)}(z_{kl}^2), \tag{14}$$

where the contour, $C(R)$, is that indicated in Fig. 2, and it is understood that on the right-hand side the point x_j is identified with x_{j+1}.

It is not difficuit to verify that the $F^{(n)}$ of the free-field case satisfy (14). One writes

$$\frac{H_1^{(1)}(m(z^2)^{\frac12})}{m(z^2)^{\frac12}} = \frac{J_1(m(z^2)^{\frac12})}{m(z^2)^{\frac12}} + i\frac{Y_1(m(z^2)^{\frac12})}{m(z^2)^{\frac12}}.$$

Now $J_1(m(z^2)^{\frac{1}{2}})/m(z^2)^{\frac{1}{2}}$ is an entire function of z^2, so that the contour integral of (14) vanishes by Cauchy's theorem. On the other hand

$$\frac{Y_1(m(z^2)^{\frac{1}{2}})}{m(z^2)^{\frac{1}{2}}}=\frac{1}{2\pi}\sum_{r=0}^{\infty}\frac{(-m^2z^2/4)^r}{r!(r+1)!}$$
$$\times\left\{2\log\left[\frac{m(z^2)^{\frac{1}{2}}}{2}\right]-\psi(r+1)-\psi(r+2)\right\}-\frac{2}{\pi m^2z^2}.$$

The precise definition of the $\psi(r)$ is irrelevant since their contribution to the integrand is an entire function. The contribution from the last term is

$$\frac{m^2}{8\pi i}\int_{C(R)}dz^2\left[\frac{-2i}{\pi m^2z^2}\right]=\frac{1}{2\pi i}.$$

From the terms involving the logarithm, we get

$$\frac{1}{2\pi}\sum_{r=0}^{\infty}\frac{(-m^2R/4)^{r+1}}{r!(r+1)!(r+1)},$$

which approaches zero in the limit $R\rightarrow 0$. The resultant formula,

$$\lim_{R\to 0}\int_{C(R)}\frac{H_1^{(1)}(m(z^2)^{\frac{1}{2}})}{m(z^2)^{\frac{1}{2}}}=\frac{1}{2\pi i}$$

shows that the $F^{(n)}$ of Eq. (9) satisfy (14).

In current renormalization theory, the canonical commutation relation (12) holds for the unrenormalized fields, but the renormalized fields are supposed to satisfy

$$\left[\frac{\partial\phi}{\partial x^0}(x),\phi(y)\right]_-\Bigg|_{x^0=y^0}=\frac{1}{Zi}\delta(\mathbf{x}-\mathbf{y}), \tag{15}$$

where $0\leq Z\leq 1$. If $Z>0$, our previous analysis requires only trivial modifications: a $1/Z$ appears as a factor on the right-hand side of (14). However, for $Z=0$, the commutator is essentially more singular than in the case of a free field. In order to understand this case, we study $F^{(2)}$ in more detail.

It is an immediate consequence of the Lorentz invariance of $G^{(2)}$, that the most general $F^{(2)}$ is of the form

$$F^{(2)}(z^2)=\int_0^{\infty}dg(m)\frac{m^2}{8\pi i}\frac{H_1^{(1)}(m(z^2)^{\frac{1}{2}})}{m(z^2)^{\frac{1}{2}}} \tag{16}$$

(apart from a constant), where dg is a weight function which will be shown to be positive in Sec. 5. It would be tempting to argue that the operation indicated in Eq. (14) can be carried out under the integral sign in (16) so that

$$1/Z=\int_0^{\infty}dg(m). \tag{17}$$

This is indeed correct if $\int_0^{\infty}dg<\infty$, but if $\int_0^{\infty}dg=\infty$, then (17) is somewhat misleading, as the following example shows.

Let

$$dg(m)=dm,\quad\text{for}\quad m\geq 0.$$

Then, using a standard integral representation of the Hankel function, we get

$$F^{(2)}(z)=\frac{1}{(2\pi)^2}\int_1^{\infty}d\xi(\xi^2-1)^{\frac{1}{2}}\int_0^{\infty}m^2dg(m)$$
$$\times\exp[-m\xi(-z^2)^{\frac{1}{2}}]$$
$$=(-z^2)^{-\frac{1}{2}}\Gamma(3)(2\pi)^{-2}\int_1^{\infty}d\xi(\xi^2-1)^{\frac{1}{2}}\xi^{-3}$$
$$=(8\pi i)^{-1}\Box[(z^2)^{-\frac{1}{2}}]. \tag{18}$$

The corresponding singularity of the commutation relation is obtained by examining the distribution:

$$(\Psi_0,[\phi(x_1),\phi(x_2)]_-\Psi_0)=\lim_{\eta\to 0}\{F^{(2)}(z)-[F^{(2)}(z)]^*\}$$
$$=\begin{cases}0 & \xi^2=(x_1-x_2)^2<0\\ \dfrac{1}{4\pi i}\Box\left(\dfrac{\epsilon(\xi)}{(\xi^2)^{\frac{1}{2}}}\right) & \xi^2\geq 0,\end{cases} \tag{19}$$

where $\epsilon(\xi)=\mathrm{sign}(\xi^0)$.

$\epsilon(\xi)(\xi^2)^{-\frac{1}{2}}$ is that very special kind of distribution, an integrable function. However, its derivatives are not functions nor can they be expressed in terms of δ functions and their derivatives except in the following rather singular way. Let f be a testing function. Then

$$\int d^4\xi\epsilon(\xi)(\xi^2)^{-\frac{1}{2}}\Box f=\lim_{\zeta\to 0}\Bigg\{-\frac{1}{\zeta^2}\int d\Omega(\xi)\xi^{\mu}\frac{\partial f}{\partial\xi^{\mu}}(\xi)\epsilon(\xi)$$
$$-\frac{1}{\zeta^2}\int d\Omega(\xi)f(\xi)\epsilon(\xi)-\int_{R\zeta}f(\xi)\epsilon(\xi)(\xi^2)^{-}d^4\xi\Bigg\} \tag{20}$$

In (20), the first and second integrals on the right-hand side are over the entire mantle of the light cone, while the third is over the four-volume later and earlier than the hyperboloids $\xi^2=\zeta^2$, $\xi^0\geq 0$ and $\xi^2=\zeta^2$, $\xi^0\leq 0$, respectively. The second integral can be written

$$\frac{1}{\zeta^2}\int d\Omega(\xi)\epsilon(\xi)f(\xi)=\frac{2}{\zeta^2}\int d^4\xi\epsilon(\xi)\delta(\xi^2)f(\xi),$$

which is just of the right type to yield the right-hand side of (15).

To obtain the commutation rules at a fixed time, one takes $f=\partial h/\partial x^0$, where h is positive and symmetrical in time, and shrinks to zero the time interval in which h is nonzero. The first term of (20) is then zero and the last two different from zero. If f is held fixed and $\zeta\rightarrow 0$, each of the last two terms becomes arbitrarily large, although their sum nearly cancels. The expression as a whole is proportional to $h(0)$ in an approximation which becomes better and better as h shrinks toward the origin of time. Thus, the right-hand side of (15) should indeed be $i^{-1}\infty\delta(\mathbf{x}-\mathbf{y})$ but it does *not* arise from a term in $(\Psi_0,[\phi(x),\phi(y)]_-\Psi_0)$ of the form $(2\pi i)^{-1}\infty\epsilon(x-y)\times\delta((x-y)^2)$ as the derivation of (17) would indicate. We conclude that the treatment of the canonical commutation rules in the case, $\int_0^\infty dg=\infty$, requires a direct consideration of the singularities of $F^{(n)}(z)$. In such a case, to replace (14) by an appropriate alternative is not difficult, but to avoid complications unessential to the present paper we defer the discussion. In particular, in Sec. 6 we consider only the local commutation rules.

5. POSITIVE DEFINITENESS CONDITIONS

Since the length of a vector is greater than or equal to zero, we have

$$\left\|\alpha_0 f_0\Psi_0+\alpha_1\int d^4x_1 f_1(x_1)\phi(x_1)\Psi_0 + \alpha_2\int\int d^4x_1 d^4x_2 f_2(x_1,x_2)\phi(x_1)\phi(x_2)\Psi_0+\cdots\right\|^2\geq 0$$

for all $\alpha_0, \alpha_1, \alpha_2,\cdots$ and all testing functions $f_1(x_1)$, $f_2(x_1,x_2),\cdots$. Therefore,

$$\sum_{i,\,j=0}\bar{\alpha}_i\alpha_j\int\cdots\int \bar{f}_i(x_1,\cdots x_i)\times F^{(i+j)}(x_i,x_{i-1},\cdots x_1,y_1,\cdots y_j)f_j(y_1,\cdots y_j)\times d^4x_1\cdots d^4x_i d^4y_1\cdots d^4y_j\geq 0. \quad (21)$$

We will refer to (21) as the positive-definiteness conditions. The simplest consequences of (21) are obtained by setting all but one of the α_i equal to zero. For example, for $\alpha_1\neq 0$, we find

$$\int \bar{f}_1(x_1)F^{(2)}(x_1-x_2)f_1(x_2)d^4x_1d^4x_2\geq 0. \quad (22)$$

It can be shown that (22) is a necessary and sufficient condition that $F^{(2)}$ be a Fourier transform of a positive measure of not too fast increase[2] (generalized Bochner theorem). For $\alpha_2\neq 0$, we have

$$\int \bar{f}_2(x_1,x_2)F^{(4)}(x_1-x_2,\,x_2-y_1,\,y_1-y_2)\times f_2(y_1,y_2)d^4x_1d^4x_2d^4y_1d^4y_2, \quad (23)$$

an inequality whose consequences have not been characterized as neatly as those of (22).

We shall not study the relations (21) in detail in this paper but we want to point out their general significance for field theory. $F^{(n)}$ is intimately connected with the S-matrix elements for those processes in which the sum of the number of ingoing and outgoing particles is n, e.g.,

$$(\Psi_0,\phi^{\text{out}}(p_1')\phi^{\text{out}}(p_2')\phi^{\text{in}}(p_1)\phi^{\text{in}}(p_2)\Psi_0)$$

is the S-matrix element for the scattering of two mesons of momenta p_1 and p_2, to produce two mesons of momenta p_1' and p_2'. The positive-definiteness conditions will therefore imply relations not only for the S-matrix elements of a given process, but also relations between the S-matrix elements of different processes. It might be argued that such relations can yield nothing more than those results arising from the conservation of probability. We do not believe that to be true. The positive-definiteness conditions deserve a detailed investigation.

6. INVERSE PROBLEM—DETERMINATION OF A THEORY FROM ITS VACUUM EXPECTATION VALUES

If one is given analytic functions $F^{(n)}(z_{ij}^2)$, $n=0$, $1, 2, \cdots$, then one can construct a theory of a scalar field, $\phi(x)$, in which the n-fold vacuum expectation value of $\phi(x)$ is just the boundary value of $F^{(n)}$, provided that the $F^{(n)}$ satisfy certain conditions, most of which have been discussed in the preceding sections. The construction will be carried out explicitly, the conditions on $F^{(n)}$ being introduced as they are needed.

First, we reconstruct the Hilbert space from the $F^{(n)}$. Certain vectors of it will be defined as conjugate linear functionals on the spaces of testing functions. Then addition, multiplication by a scalar factor and formation of the scalar product will be defined for these vectors. The full Hilbert space is then obtained by a standard completion process.

Let Ψ_0 denote the conjugate linear functional which yields for the testing function $f^{(n)}(x_1,\cdots x_n)$ the value

$$\int \bar{f}^{(n)}(x_1,\cdots x_n)d^4x_1\cdots d^4x_n F^{(n)}(x_1,\cdots x_n).$$

Let $\Psi_{g^{(m)}}$ denote the conjugate linear functional which yields for the testing function $f^{(n)}(x_1,\cdots x_n)$, the value

$$\int \bar{f}^{(n)}(x_1,\cdots x_n)F^{(n+m)}(x_n,\cdots x_1,y_1,\cdots y_m)\times g^{(m)}(y_1,\cdots y_m)d^4x_1\cdots d^4x_n d^4y_1\cdots d^4y_m.$$

The $\Psi_{g^{(m)}}$ and Ψ_0, together with the null functional 0, can be made into the basis vectors of a linear vector space if we define multiplication by a scalar and addition as follows: $\alpha\Psi_{g^{(m)}}$ is defined as $\Psi_{\alpha g^{(m)}}$, while $\Psi_{g^{(m)}}+\Psi_{h^{(m)}}$ is the conjugate linear functional which yields for the testing function $f^{(l)}(x_1,\cdots x_l)$ the value

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[2] L. Schwartz, *Theorie des Distributions* (Hermann & Cie, Paris, 1951), Vol. 2, p. 132, Theorem 18.

$$\int \bar{f}^{(l)}(x_1,\cdots x_l)F^{(l+m)}(x_l,\cdots x_1,y_1,\cdots y_m)$$
$$\times g^{(m)}(y_1,\cdots y_m)d^4x_1\cdots d^4x_l d^4y_1\cdots d^4y_m$$
$$+\int f^{(l)}(x_1,\cdots x_l)F^{(l+n)}(x_l,\cdots x_1,y_1,\cdots y_n)$$
$$\times h^{(n)}(y_1,\cdots y_n)d^4x_1\cdots d^4x_l d^4y_1\cdots d^4y_n.$$

The sum $\Psi_0+\Psi_{g^{(m)}}$ is defined similarly. It is obvious indeed that addition is associative and commutative and has a zero, the null functional. Further, inverses (negatives) exist and scalar multiplication is commutative, associative, and distributive. In short, the set of all linear combinations of the basis vectors $\Psi_{g^{(m)}}$ and Ψ_0 forms a linear vector space, $\mathfrak{H}$.

We introduce a scalar product into $\mathfrak{H}$ by means of the following formulas:

$$(\Psi_{g^{(n)}},\Psi_{h^{(m)}})=\int \bar{g}^{(n)}(x_1,\cdots x_n)F^{(n+m)}$$
$$\times(x_n,\cdots x_1,y_1,\cdots y_m)h^{(m)}(y_1,\cdots y_m)$$
$$\times d^4x_1\cdots d^4x_n d^4y_1\cdots d^4y_m, \quad (24)$$
$$(\Psi_0,\Psi_{g^{(m)}})=\int F^{(m)}(y_1,\cdots y_m)$$
$$\times g^{(m)}(y_1,\cdots y_m)d^4y_1\cdots d^4y_m. \quad (25)$$

Equations (24) and (25) define the scalar product for the basis of our linear vector space. It then turns out that for linear combinations of the basis vectors, it is consistent to define it as linear in its second argument and antilinear in its first.

The scalar product so defined will have the desirable property, $(\Phi,\Psi)=[(\Psi,\Phi)]^*$, by virtue of the hermiticity condition (5) on $F^{(n)}$. It will be positive definite by virtue of the positive definiteness conditions (21) on $F^{(n)}$. Thus, with this definition, $\mathfrak{H}$ has all the properties of a Hilbert space except completeness. So we complete it by the standard method.

Next we define the unitary operators $U(a,\Lambda)$ which give the representation of the inhomogeneous Lorentz group. Consider first the action of $U(a,\Lambda)$ for $\{a,\Lambda\}$ without time inversion on the basis elements Ψ_0 and $\Psi_{g^{(n)}}$:

$$U(a,\Lambda)\Psi_0=\Psi_0,$$
$$U(a,\Lambda)\Psi_{g^{(n)}}=\Psi_{g'^{(n)}},$$

where

$$g'^{(n)}(x_1,\cdots x_n)=g^{(n)}(\Lambda^{-1}(x_1-a),\cdots\Lambda^{-1}(x_n-a)).$$

The $U(a,\Lambda)$ preserve scalar products of the basis vectors,

$$(U(a,\Lambda)\Psi_{g^{(n)}},U(a,\Lambda)\Psi_{h^{(m)}})=(\Psi_{g^{(n)}},\Psi_{h^{(m)}}),$$

by virtue of the Lorentz invariance property of the $F^{(n)}$ given in Eq. (2). Furthermore, on the basis vectors,

$$U(a,\Lambda)U(b,\mathrm{M})=U(a+\Lambda b,\Lambda\mathrm{M}). \quad (26)$$

The operators $U(a,\Lambda)$ can be extended to all vectors in $\mathfrak{H}$ by linearity and, so defined, are unitary.

For the $\{a,\Lambda\}$ containing time inversion we define

$$U(a,\Lambda)\Psi_0=\Psi_0,$$
$$U(a,\Lambda)\Psi_{g^{(n)}}=\Psi_{g^{\times(n)}}, \quad (27)$$

where

$$g^{\times(n)}(x_1,\cdots x_n)=[g^{(n)}(\Lambda^{-1}(x_1-a),\cdots\Lambda^{-1}(x_n-a))]^*.$$

This definition has as a consequence

$$(U(a,\Lambda)\Psi_{g^{(n)}},U(a,\Lambda)\Psi_{h^{(m)}})=[(\Psi_{g^{(n)}},\Psi_{h^{(m)}})]^*,$$

by virtue of the Lorentz invariance property of the $F^{(n)}$ given in Eq. (4). Just as for the operators for $\{a,\Lambda\}$ without time inversion, (26) is satisfied, and the $U(a,\Lambda)$ can be extended to all vectors, and, so defined, are antiunitary.

The $U(a,\Lambda)$ which are determined in this way are guaranteed to contain no negative energies by virtue of the analyticity of the $F^{(n)}$ in the future tube. The details of their momentum spectrum can be ascertained from the set on which the Fourier transforms of the $F^{(n)}$ do not vanish. The real four-vector, p, is in the momentum spectrum of $U(a,\Lambda)$ if for some $F^{(n)}$, the Fourier transform with respect to at least one of the variables does not vanish at p.

The action of the field operator $\phi(x)$ [or better its average, $\phi(f)$, with the testing function $f(y)$] on the vector $\Psi_{g^{(n)}}$ is defined as follows: $\phi(f)\Psi_{g^{(m)}}$ is the conjugate linear functional which, for the testing function $f^{(n)}(x_1,\cdots x_n)$ yields the number

$$\int f^{(n)}(x_1,\cdots x_n)F^{(n+m+1)}(x_n,\cdots x_1,y,y_1,\cdots y_m)$$
$$\times f(y)g^{(m)}(y_1,\cdots y_m)d^4x_1\cdots d^4x_n d^4y_1\cdots d^4y_m.$$

This is just the vector $\Psi_{fg^{(m)}}$, where $f(y)g^{(m)}(y_1,\cdots y_m)$ is regarded as a testing function in $(m+1)$ variables. $\phi(f)\Psi_0$ is defined as Ψ_f. $\phi(f)$ is defined as the closed linear extension of this operator.

On the basis vectors and their finite linear combinations $\phi(f)$ possesses the property

$$\phi(f)^*=\phi(\bar{f}),$$

where $\bar{f}(x)$ is the complex conjugate of $f(x)$. Thus, $\phi(f)$ will be formally Hermitean if f is real.

The transformation properties of the operator $\phi(f)$ follow immediately from those of the $\Psi_{g^{(n)}}$. For $U(a,\Lambda)$ unitary, we have

$$U(a,\Lambda)\phi(f)\Psi_{g^{(n)}}=U(a,\Lambda)\Psi_{fg^{(n)}}$$
$$=\Psi_{f'g'^{(n)}}=\phi(f')U(a,\Lambda)\Psi_{g^{(n)}}$$

Therefore,

$$U(a,\Lambda)\phi(f)=\phi(f')U(a,\Lambda),$$

with $f'(x)=f[\Lambda^{-1}(x-a)]$ in agreement with (1). For the anti-unitary $U(a,\Lambda)$ we have similarly:

$$\begin{aligned} U(a,\Lambda)\phi(f)\Psi_{g^{(n)}} &= U(a,\Lambda)\Psi_{fg^{(n)}} \\ &= \Psi_{f^\times g^{\times(n)}} = \phi(f^\times)U(a,\Lambda)\Psi_{g^{(n)}} \end{aligned}$$

Therefore,

$$U(a,\Lambda)\phi(f)=\phi(f^\times)U(a,\Lambda),$$

with $f^\times$ defined as in (27) in agreement with (1).

The local commutation rules of $\phi(f)$ can be verified as follows:

$$\phi(f_1)\phi(f_2)\Psi_{g^{(n)}}=\Psi_{f_1f_2g^{(n)}}=\Psi_{f_2f_1g^{(n)}}=\phi(f_2)\phi(f_1)\Psi_{g^{(n)}},$$

when $f_1(x)f_2(y)$ for all $(x-y)^2\geq 0$. The proof depends on the fact that for such f_1 and f_2,

$$f_1(x)f_2(y)[F^{(\cdots)}(\cdots x,y,\cdots)-F^{(\cdots)}(\cdots y,x,\cdots)]=0,$$

by virtue of Eq. (10).

To complete the reconstruction of the theory we need only show that the vacuum expectation values of products of $\phi(f)$'s are the $F^{(n)}$. This is an easy consequence of the formula $\phi(f_1)\phi(f_2)\cdots\phi(f_n)\Psi_0=\Psi_{f_1\cdots f_n}$. It implies

$$(\Psi_0,\phi(f_1)\cdots\phi(f_n)\Psi_0) = \int f_1(x_1)\cdots f_n(x_n)F^{(n)}(x_1,\cdots x_n)d^4x_1\cdots d^4x_n,$$

which was to be proved.

Our reconstruction remains valid in a theory in which the field $\phi(x)$ is not a complete description of the system, e.g., in a theory of interacting neutral mesons and nucleons. However, in such a case the reconstruction process given here will not recover the entire Hilbert space. If one were to define a theory by its analytic functions $F^{(n)}$, rather than by its field equations and commutation rules, then, to be sure that the theory was one of a single field $\phi(x)$, one would have to impose some kind of "completeness" requirement. For example one could require that the set of vectors $\phi(f_1)\cdots\phi(f_n)\Psi_0$ for $n=0, 1, 2, \cdots$ span the whole Hilbert space, where the f_j are testing functions which vanish outside of a spacelike slice of space time of arbitrarily small thickness Δt in the time direction.

7. CONCLUSION

A theory of a neutral scalar field can be reformulated as a theory of a denumerable set of analytic functions of complex variables, $F^{(n)}$, $n=0, 1, 2, \cdots$. Relativistic invariance implies that the $F^{(n)}$ are invariant under Lorentz transformation without time inversion and are therefore functions of certain complex variables z_{ij}^2. Local commutation rules of $\phi(x)$ imply $F^{(n)}(z_{ij}^2) =F^{(n)}(Pz_{ij}^2)$, where P is any of a certain set of permutations of the labels i, j; the positive definiteness of the scalar product implies a set of inequalities connecting the boundary values of the $F^{(n)}$. Given a set of $F^{(n)}$ satisfying the conditions listed, one can reconstruct a theory of a neutral scalar field.

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ON QUANTUM FIELD THEORIES

BY

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In the present paper[1], the general conditions which a relativistic quantum theory of interacting particles must satisfy are investigated and brought into mathematical form. Some difficulties connected with the infinite number of degrees of freedom are pointed out. Especially the fact that the canonical commutation relations no longer have unique solutions must be taken into account in all discussions of field theory. It is shown that the "free field vacuum" of the Tamm-Dancoff method and Dyson's matrix $U(t_1, t_2)$ for finite t_1 or t_2 cannot exist. The possibility of a conventional field theory in which the fields commute for equal times is investigated.

Introduction.

In the past few years, considerable effort has been devoted to the question whether a field theory of conventional type plus renormalization may be regarded as a well defined mathematical scheme which is only formulated in a somewhat awkward way. An affirmative answer to this question would mean that it is possible—at least in a certain idealization—to separate the problem of the interaction of elementary particles from that of their existence and constitution. Of course it is quite clear that such a separation cannot ultimately be satisfactory[2]. From a certain point of view, it might even be correct to say that it is just this

[1] The major part of this work has been done during the author's stay at the Institute for Theoretical Physics in Copenhagen as a member of the CERN Theoretical Study Group.

[2] To mention only one of the many arguments which have been put forward in this connection: the presence of electromagnetic interaction is always accompanied by a mass contribution of the order of magnitude of the electron mass (mass difference proton-neutron, charged and neutral meson). This seems to suggest that, in a more satisfactory formulation of electrodynamics, one should not have the "mechanical mass" of the electron as the basic constant, but something independent of the particle type, say a cut-off length l. Ordinary electrodynamics corresponds to the idealization $l = 0$, in which case the mass becomes infinite.

undue separation which causes the divergence difficulties in field theory. On the other hand, we must keep in mind that, in essence, quantum field theory is (or rather should be) nothing but an extension of quantum mechanics to a system with infinitely many degrees of freedom in such a way that, instead of invariance under Galilei transformations, we have Lorentz invariance. Now it is evident[3] that there is no contradiction between the general framework of quantum physics, the relativity principle, and the fact that we want to deal with a system of arbitrarily many particles. Therefore a mathematically consistent formulation of such a theory should certainly be possible without introducing essentially new physical ideas to explain the constitution of elementary particles. This statement contrasts strangely with the fact that it is extremely easy to write down any number of formalisms which may be regarded as the quantum mechanics of some hypothetical system, whereas up to now not a single model of a field theory has been given which meets all consistency requirements. The aim of the present paper is to work out some criteria of consistency to which the formalism must be subjected, to point out a few of the mathematical difficulties connected with the infinite number of degrees of freedom, and to apply the results to a discussion of some of the basic assumptions and calculating methods of a field theory of conventional type. In this way we come back to the question mentioned in the beginning.

In Chapter I, the general assumptions are formulated and combined in a form suitable for later application. They include the principles of quantum physics, relativistic invariance, and the "particle postulates", i. e. those requirements which ensure that the theory can be interpreted as describing the interaction processes of particles. Chapter II deals with some mathematical questions. A few simple consequences, which may be drawn without recourse to perturbation calculations, are given in the first paragraphs of Chapter III. It is shown that general field theories exist, i. e. theories in which the commutation relations of the field operators for equal times are not fixed a priori, and that they are just as wide in scope as are S-matrix theories. For the case of conventional theories in which the field operators are required to commute for equal times, a very simple argument shows that the so-called "free field vacuum" does not exist. This implies that in the (old) Tamm-Dancoff method, in which this concept plays an important role, additional difficulties are introduced, and also that Dyson's matrix $U(t_1, t_2)$ cannot exist if one or both of the times t_1, t_2 are finite. The question as to whether a conventional theory is possible at all is investigated in the last paragraph by a perturbation calculation. It appears that the requirement of commutability for equal times leaves only a few types of models, and also that the appearance of highly singular functions, which need careful definition and handling, is an inevitable consequence at least in perturbation calculations.

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[3] This will also be discussed in Chapter I.

In order to eliminate all difficulties or complications which are not essential to the main question, the discussion has been limited, in Chapter III, to the simplest conceivable physical situation, the case in which only one type of particle exists. It is supposed to have spin zero and mass m.

I. General assumptions and immediate consequences.

§ 1. The postulates.

A. Quantum Physics.

We deal with a Hilbert space $\mathfrak{h}$. To every "state" of the system corresponds a normalized vector Ψ of $\mathfrak{h}$. If we know that the system is in the state Ψ_1, the probability of observing a state Ψ_2 is given by the absolute square of the scalar product

$$w = |(\Psi_2, \Psi_1)|^2.$$

To connect the abstract scheme to physical reality it is necessary to give the coordination between the "states" experimentally defined in terms of Geiger counters etc., and their corresponding elements Ψ of $\mathfrak{h}$. This is called "the physical interpretation of the theory".

B. Relativistic Invariance.

The term Lorentz transformation (L. T.) will be understood to include also translation in space and time, but not reflections; that is to say, we deal with the 10-parametric inhomogeneous Lorentz group where the elements may be characterized by the coefficients $a_{\mu\nu}$, b_μ of the transformation

$$x'_\mu = a_{\mu\nu}\, x_\nu - b_\mu \qquad (\mu, \nu = 0,\ 1,\ 2,\ 3).$$

The index 0 refers to the (real) time coordinate. As the scalar product between two 4-vectors p and x we shall write

$$px = p_i\, x_i - p_0 x_0 \qquad (i = 1, 2, 3).$$

We also consider the reflection elements s_r, s_t and $s = s_r\, s_t$ giving the transformations

$$\begin{array}{lll} s_r: & x'_i = -x_i, & x'_0 = x_0 \\ s_t: & x'_i = x_i, & x'_0 = -x_0 \\ s\ : & x'_\mu = -x_\mu. & \end{array}$$

The invariance of the theory with respect to these reflections will not be relevant in the context of this paper but, for the sake of completeness, a short discussion is included.

It is convenient to state the invariance requirements in connection with A in a form introduced by WIGNER[4]: To every element L of the Lorentz group there is attached a unitary operator $D(L)$ in $\mathfrak{h}$ so that

$$D(L_1)\, D(L_2) = \pm\, D(L_1 L_2).$$

Mathematically speaking: in $\mathfrak{h}$ we must have a representation of the Lorentz group. The physical significance of the operators $D(L)$ is the following. Let M_Ψ be the apparatus needed for the experimental realization of the state Ψ. We can envisage another apparatus $M_{\Psi'}$ which—if viewed from a Lorentz frame Σ'—would be described in exactly the same way as M_Ψ is described by an observer in Σ, i. e. $M_{\Psi'}$ differs from M_{Ψ} only in space-time orientation. Then, if L is the L. T. which carries Σ' over into Σ, we have

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[4] E. WIGNER, Ann. of Math. 40, 149 (1939). See also V. BARGMAN and E. WIGNER, Proc. Nat. Acad. Sci. 34, 211 (1948).

$$\Psi' = D(L)\,\Psi$$

for every state Ψ.

The invariance against reflections may be formulated in a similar way. One difference is that the operators T, C, and J, which correspond to the elements s_r, s_t and s, respectively, could also be "anti-unitary"[5]. In fact, it appears that, for the cases of physical interest, C and J are anti-unitary whereas T is unitary. The physical significance of T is straightforward. We have to consider two experimental arrangements which are related like a right- and left-hand glove. In the case of C, a similar general definition does not seem possible since we cannot interchange past and future. However, for the special case of a state which is localized around a point $x = (x, t)$, one may take the time-inverted state as the state of localization around $x' = (x, -t)$.

It is important to keep in mind that we can speak of the invariance of the theory with respect to these transformations only if operators T, C, J exist which have both the correct structure relations with each other and with the $D(L)$,[6] and also the physical significance mentioned. One can show, for instance, that in all representations of the Lorentz group, which correspond to particle systems, there exist operators which obey the structure relations prescribed for T or C.[7] It is not true, however, that every Lorentz invariant theory is automatically invariant under space or time inversion, since the formal operators T, C need not have the specified observational significance if more than one particle is involved[8].

C. Particle Postulates.

We shall not here distinguish between "elementary" and "compound" particles. This is a matter of convenience rather than of principle. As characteristic of a particle we regard the localization of the events which may be caused by it. Thus, we could use, for instance, a coincidence arrangement of two Geiger counters separated by a distance d with a resolving time $\delta t \langle\langle \frac{d}{c}$ to single out experimentally the states containing only one particle. If we consider all such coincidence arrangements for arbitrary position, orientation, and velocity of the apparatus and arbitrary distance d (which is only supposed to be larger than some d_{min}), then the one-particle states are those which give a negative result in all these measurements[9]. The magnitude of d_{min} is irrelevant. It could be taken as large as one pleases but, for practical reasons, it will be convenient to set an upper limit to the size of the object which we choose to call a particle.

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[5] An anti-unitary operator is an operator which transforms a linear relation $\Sigma a_k \Psi_k = 0$ into the linear relation between the image vectors $\Sigma a_k^* \Psi_k' = 0$ (complex conjugate coefficients!). The scalar product of the image vectors is the complex conjugate of the scalar product of the original vectors $(\Psi_1' \Psi_2') = (\Psi_1 \Psi_2)^*$.

[6] For instance, $J\,D(b) = D(-b)\,J$, $J\,D(\Lambda) = D(\Lambda)\,J$, $J^2 = \pm 1$, where b is a translation, Λ a homogeneous L. T.

[7] The case of particles with mass zero is an exception. The statement continues to hold, however, if $m = 0$ is regarded as the limiting case of a small mass.

[8] I had overlooked this fact in the manuscripts mentioned in ref. 23. The point was cleared up in a discussion with G. LÜDERS and L. L. FOLDY.

For a theoretical analysis, it is important that the manifold of one-particle states is thus defined as a relativistically invariant subspace $\mathfrak{h}^{(1)}$ of $\mathfrak{h}$. We can now split $\mathfrak{h}^{(1)}$ further into invariant subspaces $\mathfrak{h}_j^{(1)}$ and imagine this process to be carried through as far as possible, that is to say, so far that each $\mathfrak{h}_j^{(1)}$ belongs to an *irreducible* representation of the Lorentz group. The different $\mathfrak{h}_j^{(1)}$ can be distinguished by attributes which have a relativistically invariant meaning. These attributes can be considered as the attributes of the "particle species j". It is therefore just a definition of the concept "particle species" if we say

C_1. *The manifold of states of one particle of type j belongs to an irreducible representation $D_j(L)$ of the Lorentz group.*

The possible irreducible representations have been classified by WIGNER[4]. Each of them is characterized by two numbers m and s. These have the physical meaning of the mass and the spin of the particle, respectively.

The representations belonging to imaginary values of m can obviously not be attached to particles and, therefore, the first action of the particle postulates must be to exclude these representations. Then, for $m > 0$, the spin can take only integer or half integer values and the representations may be described within the well-known formalism of the wave equations of spin particles. For $m = 0$ there exist also representations corresponding to a continuous spin variable. Whether or not these representations have physical importance is yet an open question. They will not be considered here.

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[9] The vacuum state is of course also admitted, but can be excluded in a trivial way by single counter measurements.

The decomposition of $\mathfrak{h}^{(1)}$ into the $\mathfrak{h}_j^{(1)}$ becomes ambiguous if $\mathfrak{h}^{(1)}$ contains several equivalent irreducible representations, i. e. if we have several different particles with equal mass and spin. This is analogous to the appearance of degenerate eigenfunctions in an eigenvalue problem. Incidentally, it may be noted that the eigenvalue problem for the Hamiltonian in conventional quantum mechanics is in field theory replaced by the mathematical problem of reducing the representation $D(L)$ into its irreducible components. In the degenerate case, it may be more appropriate to consider the total space $\Sigma \mathfrak{h}_j^{(1)}$ spanned by all the representations which belong to the same mass and spin as the manifold of states for *one* particle. This would, for instance, mean that instead of protons and neutrons we speak of nucleons, etc.

Now going over to situations in which more than one particle is involved, it is convenient to introduce the concept of "partial state around a point x". By this we understand those properties of the system which may be measured by any experimental set-up within a large but finite space-time volume around x. Then, we formulate the following fundamental assumption:

C_2. *Whatever total state of the system we consider, the partial state around $x = (x, t)$ approaches a one-particle state for $|t| \to \infty$.*

For this assumption three conditions are necessary. First, we exclude from consideration situations in which the particles fill the total space with a finite density. Secondly, there are (per definitionem) no "bound states". Thirdly, in this general discussion we allow only normalizable state vectors (proper elements of $\mathfrak{h}$). This ensures that we have only a finite interaction region outside which the particles are separated so much that the attraction between them can be neglected. It does not mean of course that in practical calculations we could not use plane waves as well, if only they are handled with proper care.

Now, the experimental characterization of an arbitrary state Ψ may be given in terms of measuring results obtained in the distant past, for short at $t = -\infty$. As the particles may then be considered as isolated, these results can be described by means of single-particle states. If we introduce in $\mathfrak{h}_j^{(1)}$ an arbitrary basis system $f_k^{(j)}$, a complete basis in the total $\mathfrak{h}$ is given by the vectors $\bar{\Phi}^{(N)}_{\substack{jj'\cdots\\kl\cdots}}$. This symbol describes a state which asymptotically at $t = -\infty$ (expressed by the minus sign above Φ) has N particles of types $j, j' \ldots$ which are in the single-particle states $k, l \ldots$ etc.

If in $\Phi^{(N)}$ one particle species occurs several times, we must take the symmetry principle into account.

C_3. Each particle obeys either Bose- or Fermi statistics.

$$\bar{\Phi}_{\substack{jj\\kl}} = \begin{cases} +\bar{\Phi}_{\substack{jj\\lk}} & \textit{for Bose particles}\\[1ex] -\bar{\Phi}_{\substack{jj\\lk}} & \textit{for Fermi particles}\end{cases}$$

The operators $D(L)$ take a simple form in this basis system. Each single particle state $f_k^{(j)}$ entering in $\Phi^{(N)}$ is transformed independently. Thus, within the manifold of asymptotic two-particle states $\bar{\Phi}^{(2)}_{(jj')}$, the effective part of $D(L)$ is just the Kronecker product $D_j(L) \times D_{j'}(L)$, where $D_j(L)$ and $D_{j'}(L)$ are the *irreducible* representations attached to the particle types j and j', respectively. If $j = j'$, we must again take account of the symmetry principle. We indicate the effect of symmetrization by brackets and in this case write $\{D_j(L) \times D_j(L)\}$. The structure of the total representation in $\mathfrak{h}$ may then be written

$$\left.\begin{aligned} D(L) = 1 + \sum_j D_j(L) + \sum_j \{D_j(L) \times D_j(L)\} + \cdots \\ + \sum_{j,j'} D_j(L) \times D_{j'}(L) + \cdots . \end{aligned}\right\} \tag{1}$$

The meaning of the + and Σ signs is that $\mathfrak{h}$ may be decomposed into orthogonal subspaces (namely the asymptotic $0, 1, 2 \ldots$ particle states) in each of which one of the terms is operating. The identical transformation (1) refers to the vacuum state. We have thus already incorporated the further postulate

C_4. There is one state in $\mathfrak{h}$ which is invariant under all L. T., the vacuum state Φ_0.

Equation (1) is important because its meaning is not restricted to the special basis system $\bar{\Phi}^{(N)}$ from which we started. The (direct) sum and direct product of representations have an invariant group theoretical meaning. Thus, the particle postulates fix the structure of the representation $D(L)$ completely. To give an example: if we work in the center of mass system (spatial momentum zero), then the energy spectrum must have a characteristic structure: On top of a number of discrete eigenvalues $0, m_j$, we have a continuous spectrum starting at $2\, m_1$ and having a branch point at each value $\Sigma n_j\, m_j$, where the n_j are an arbitrary set of integer numbers. Similar considerations can be made for the angular momentum. Generally speaking, if in a theory the operators P_μ and $M_{\mu\nu}$ of linear and angular momentum (infinitesimal translations and rotations) are given, then (1) affords a mathematically clean-cut criterion as to whether the theory is acceptable on the basis of the postulates C. Since this decision is essentially a problem of group theory, it can be hoped that some more powerful methods of attack will be developed than those available to physicists at present.

It may be instructive to illustrate the argument leading to (1) by an example from wave mechanics. This may make the significance of the basis states $\Phi^{(N)}$ somewhat clearer and also give an exact meaning to statements like "at $t = \infty$ the interaction vanishes". Let us consider a non-relativistic two-body problem without bound states. The Hamiltonian shall be

$$H = -\frac{1}{2\,m}(\nabla_1^2 + \nabla_2^2) + V(|\vec{x}_1 - \vec{x}_2|) = H_0 + V.$$

Ψ is an arbitrary state vector. As always in this paper, the Heisenberg picture will be used. Ψ is represented by a function $\psi(\vec{x}_1, \vec{x}_2)$ which is interpreted as the probability amplitude for the positions $\vec{x}_1, \vec{x}_2$ of the particles at $t = 0$. Now we consider the sequence of unitary operators

$$U(t) = e^{tH_0t}\, e^{-iHt}. \tag{2}$$

It may be shown that, if V decreases more strongly than $\frac{1}{r}$ with the separation of the particles, then this sequence of operators converges strongly for $|t| \rightarrow \infty$[10,11]. The limit for $t = -\infty$ we call R. From the definition (2) it follows that

$$U(t)\, e^{iHt'} \Psi = e^{iH_0 t'} U(t - t')\, \Psi. \tag{3}$$

For $t = -\infty$

$$R e^{iHt'} \Psi = e^{iH_0 t'} R \Psi. \tag{4}$$

If we now introduce a new basis system in such a way that Ψ is represented by the wave function $\varphi(\vec{x}_1, \vec{x}_2)$ which belonged to the state R, Ψ in the old basis, then the (total) Hamiltonian takes the simple form $H = -\frac{1}{2m}(\nabla_1^2 + \nabla_2^2)$. $\varphi(\vec{x}_1, \vec{x}_2)$ is now the probability amplitude for the positions $\vec{x}_1, \vec{x}_2$ at $t = 0$ as it would be calculated from the results of asymptotic observations under the assumption that the particles had moved without interaction until $t = 0$. This new basis system is the analogue to the $\Phi^{(N)}$, apart from the fact that there we have expanded according to a discrete set of eigenfunctions. In the new basis, H is separable into one-particle components, which corresponds to the independent transformation of the $f_k^{(j)}$ in $\Phi^{(N)}$. In this example, H is regarded as the prototype for all Lorentz transformations. The "vanishing of the interaction" at $|t| = \infty$ is to be understood in the sense of a strong operator convergence. This means that the operator of the interaction energy *does not itself vanish*, but that, applied to a state $e^{iHt}\Psi$ (where Ψ is arbitrary, but fixed), it gives an image vector which has zero length in the limit $|t| \rightarrow \infty$. These remarks may serve to handle the limits $|t| \rightarrow \infty$ in the time-dependent formulation of scattering problems in a way which is both nearer to the physical meaning and simpler in mathematics than the introduction of a convergence factor $e^{-\alpha|t|}$ to "switch off the interaction". One must only keep in mind that all the asymptotic relations are to be regarded in the sense of a strong operator convergence.

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[10] This theorem provides the basis for the treatment of scattering problems by Dirac's time-dependent perturbation method or, in more modern language, by means of the interaction representation.

[11] A sequence of operators U_k is said to converge strongly towards a limit U if the application of $(U_k - U)$ on an *arbitrary* but fixed state Ψ produces a sequence of image vectors Ψ'_k which decrease in length towards zero for $k \rightarrow \infty$:

$$\lim_{k\rightarrow\infty} \|(U_k - U), \Psi\| = 0.$$

§ 2. The *S*-matrix.

The same arguments which have been used in § 1 to define the basis $\overset{-}{\Phi}^{(N)}$ can be carried through if everywhere we put $t = +\infty$ instead of $t = -\infty$. Then we obtain a second complete orthogonal basis $\overset{+}{\Phi}^{(N)}$. The unitary operator which connects both of them is the *S*-matrix

$$\overset{-}{\Phi}_{\xi} = S\overset{+}{\Phi}_{\xi}. \tag{5}$$

Here, ξ is the array of indices (identical on both sides of the equation) which characterizes the results of the asymptotic observations. We simply call ξ the "configuration".

$$S_{\xi'\xi} = \langle \overset{+}{\Phi}_{\xi'} \,|\, S \,|\, \overset{+}{\Phi}_{\xi} \rangle = \langle \overset{-}{\Phi}_{\xi'} \,|\, S \,|\, \overset{-}{\Phi}_{\xi} \rangle \tag{6}$$

is the transition amplitude from the asymptotic configuration ξ at $t = -\infty$ to ξ' at $t = +\infty$. These matrix elements are therefore related in the well-known way to the cross sections for various processes. Because an L. T. changes the configuration at $t = +\infty$ and $t = -\infty$ in the same way, we have

$$[D(L), S] = 0 \qquad \text{for every } L \tag{7}$$

and similarly

$$[T, S] = 0. \tag{7a}$$

In the case of time inversion, we first define the operators $\overset{-}{C}$ and $\overset{+}{C}$ by

$$\overset{-}{C}\overset{-}{\Phi}_{\xi} = \overset{-}{\Phi}_{\check{\xi}}; \quad \overset{+}{C}\overset{+}{\Phi}_{\xi} = \overset{+}{\Phi}_{\check{\xi}}, \tag{8}$$

where $\check{\xi}$ is obtained from ξ by applying the single-particle time inversion operators C_j to all the one-particle states $f_k^{(j)}$ entering into the configuration. The actual time inversion operator C must, however, not only change ξ into $\check{\xi}$, but also interchange $t = +\infty$ and $t = -\infty$, i. e.

$$C\overset{-}{\Phi}_{\xi} = \overset{+}{\Phi}_{\check{\xi}} \tag{9}$$

or

$$CS = \overset{+}{C};\ SC = \overset{-}{C}. \tag{9a}$$

Now

$$C^2 = \overset{-}{C}{}^2 = \overset{+}{C}{}^2 = \pm 1.$$

Therefore

$$S^\dagger \overline{C} = \overline{C} S^\dagger \tag{10}$$

and, similarly,

$$S^\dagger \overline{J} = \overline{J} S. \tag{10a}$$

This is a general formulation of the principle of detailed balancing[12].

The S-matrix defined by (5) would perhaps better be called the complete S-matrix. In the majority of the work on S-matrix theory, a distinction between elementary particles and bound states is made, and the term S-matrix refers to the submatrix of our S between the scattering states of the elementary particles.

§ 3. The formalism of field theory.

A complete operator system[13] in ζ is given by the creation and destruction operators of the various particles in the different quantum states. They will be called $\bar{u}_k^{(j)}$ and $\bar{u}_k^{(j)\dagger}$, respectively, if they refer to the asymptotic configuration at $t = -\infty$. $\bar{u}_k^{(j)\dagger}$ acts on a basis vector by adding to the configuration ξ one particle of type j in the state k:

$$\bar{u}_k^{(j)\dagger} \overline{\Phi}_\xi = \lambda \overline{\Phi}_{\xi + \binom{j}{k}}. \tag{11}$$

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[12] It is interesting to see at what point of the argument leading to (10) the requirement of invariance under time inversion is used. $\overset{-}{C}$ and $\overset{+}{C}$ may always be defined in a Lorentz invariant theory and they have the correct structure relations. The definition (9a) is adapted to give the correct physical significance to C. Thus, if the theory is invariant under time inversion, C must also have the correct structure relations, and vice versa. Now the relations with $D(L)$ are correct, because S commutes with all $D(L)$. However, $C^2 = \pm 1$ is not obvious and this imposes some restrictions on S (detailed balancing).

[13] "Complete" means that every operator in ζ may be approximated by polynomials in these basic operators.

The corresponding destruction operator is the hermitian conjugate. It is convenient to fix the factor λ so that the operators obey the canonical commutation relations[14]

$$\left.\begin{array}{l} [\bar{u}_k^{(j)}\,\bar{u}_l^{(j')}] = [\bar{u}_k^{(j)\dagger}\,\bar{u}_l^{(j')\dagger}] = 0 \\ [\bar{u}_k^{(j)}\,\bar{u}_l^{(j')\dagger}] = \delta_{kl}\delta_{jj'} \end{array}\right\}\text{ for Bose particles, }\Bigg\} \quad (12)$$

$$\left.\begin{array}{l} \{\bar{u}_k^{(j)}\,\bar{u}_l^{(j')}\} = \{\bar{u}_k^{(j)\dagger}\,\bar{u}_l^{(j')\dagger}\} = 0 \\ \{\bar{u}_k^{(j)}\,u_l^{(j')\dagger}\} = \delta_{kl}\delta_{jj'}. \end{array}\right\}\text{ for Fermi particles. }\Bigg\} \quad (13)$$

If we were dealing with a system with a finite number of degrees of freedom these commutation relations alone would fix the operators $u_k^{(j)}, u_k^{(j)\dagger}$ uniquely up to a unitary transformation. In the following chapter it will be discussed that this is no longer true in our case. In order to achieve a unique specification (apart from equivalence) one must require, in addition to (12) or (13), that there exists one state (the vacuum) for which

$$\bar{u}_k^{(j)}\Phi_0 = 0 \qquad (14)$$

for all k and j.

In mathematical language: $\mathfrak{h}$ is the space of an irreducible representation of the commutator ring (12), (13), with the auxiliary condition (14); irreducible, because the operators in question form a complete system.

Again we can replace the minus signs above Φ and $u^\dagger$ in (11) by plus signs and obtain a second set of creation and destruction operators referring now to the asymptotic configurations at $t = +\infty$. The two sets are of course connected by the S-matrix.

The creation and destruction operators $\bar{u}_k^{(j)\dagger}, \bar{u}_k^{(j)}$ can be combined in the well-known way to a continuous manifold of operators $\overset{-}{\varphi}{}^{(j)}(x)$ (see, for instance, Chapter III, § 1). These operator fields are the "incoming fields" of YANG-FELDMAN and KÄLLÉN[15]; similarly, the $\overset{+}{\varphi}{}^{(j)}(x)$ are the "outgoing fields". The essential feature of field theory is that these operators $\overset{-}{\varphi}^{(j)}(x)$ and $\overset{+}{\varphi}^{(j)}(x)$, which have a simple physical interpretation in terms of asymptotic observations, are regarded as the asymptotic limits for $t = \pm\infty$ of other operators, the "actual fields" $\psi^{(j)}(x)$. In symbols

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[14] The deeper significance of this choice is that the creation operators in different basis systems of one-particle states are simply related. Thus, if $f_k = \Sigma C_{k\alpha} f_\alpha$, then $u_k^\dagger = \Sigma C_{k\alpha} u_\alpha^\dagger$.

[15] C. N. YANG and D. FELDMAN, Phys. Rev. **79**, 972 (1950); G. KÄLLÉN, Ark. f. Fys. **2**, 187 (1951).

$$\psi^{(j)}(x) \rightarrow \overset{\pm}{\varphi}_j(x) \quad \text{for} \quad t \rightarrow \pm\infty, \tag{15}$$

where the arrow again indicates strong operator convergence. These $\psi^j(x)$ are regarded as the basic quantities of the theory. They are defined by their commutation relations (at least in theories of conventional type), and the infinitesimal Lorentz operators P_μ and $M_{\mu\nu}$ are given as functions of them. This procedure is in complete analogy to that in quantum mechanics. There is only one additional requirement in field theory, namely, that the $\psi^j(x)$ should have the simple relativistic transformation properties of a field. The effect is to fix the $M_{\mu\nu}$ as soon as the P_μ are given. This restriction is, however, not very serious in itself and in fact it will be explicitly demonstrated later that theories exist which are built on the concept of covariant operator fields $\psi_j(x)$ satisfying the "asymptotic condition" (15) and which are able to yield any S-matrix. The central question is then by what relations the $\psi_j(x)$ may be defined in the basic equations of the theory. Is it possible, for instance, to require that the field operators at points separated by a space-like distance commute? Although this mathematical question will not be decided in this paper, we can at least exhibit some of the pitfalls which tend to make rather inconclusive many general statements reached by the standard methods.

In the scheme of field theory outlined above we have one field for every existing particle species. The extension to the case in which one introduces a smaller number of fields and thereby a distinction between elementary and compound particles is of course more interesting for practical applications (meson theory), but will not be considered here.

II. Mathematical Considerations.

§ 1. Inequivalent representations of the canonical commutator ring.

This section refers both to the commutation relations (12) and the anticommutation relations (13); nevertheless the discussion will be restricted to the former. Combining the indices k and j to a single index k, the relations are brought into the canonical form by substituting

$$q_k = \frac{1}{\sqrt{2}}(u_k + u_k^\dagger), \quad p_k = \frac{1}{i\sqrt{2}}(u_k - u_k^\dagger); \tag{16}$$

q_k and p_k are then hermitian operators satisfying

$$[q_k q_l] = [p_k p_l] = 0, \quad [p_k q_l] = -i\delta_{kl}. \tag{17}$$

In the conventional field theories we have commutation relations of the form

$$\left.\begin{aligned} [\psi(\vec{x})\,\psi(\vec{x}')] = [\pi(\vec{x})\,\pi(\vec{x}')] = 0, \\ [\pi(\vec{x})\,\psi(\vec{x}')] = -i\delta(\vec{x}-\vec{x}'). \end{aligned}\right\} \tag{18}$$

These are reduced to (17), for instance, by putting

$$q_k = \int f_k(\vec{x})\,\psi(\vec{x})\,d\vec{x}, \quad p_k = \int f_k(\vec{x})\,\pi(\vec{x})\,d\vec{x}, \tag{19}$$

where the $f_k(\vec{x})$ are an arbitrary complete system of real, orthogonal, normalized functions.

It is a well known fact that for a *finite number* N of degrees of freedom there is only one (irreducible) representation of the operators p, q of (17) (apart from equivalences). This may be obtained by considering the q_k as multiplication operators, the p_k as differentiation operators. It is more convenient for our purpose to work with the u_k, $u_k^\dagger$ and the "occupation numbers"

$$\nu_k = u_k^\dagger u_k \tag{20}$$

which have integer eigenvalues ranging from 0 to ∞. A complete orthogonal system is given by the simultaneous eigenfunctions of the ν_k. Each basis vector is then specified by an array of N occupation numbers $(\nu_1 \cdots \nu_N)$. u_k acts as a destruction operator, $u_k^\dagger$ as creation operator for the k^{th} "oscillator", i. e. they decrease or increase ν_k by 1. The "vacuum state" $\Phi_0 = (0, 0 \cdots 0)$ may be defined abstractly by

$$u_k \Phi_0 = 0 \qquad \text{for all } k. \tag{21}$$

If we pass now to the limit $N \rightarrow \infty$ one new feature appears. A possible basis vector results from any distribution of integer numbers ν_k over the infinitely many oscillators. The "number" of these possibilities is *no longer countable.* It is given by $\aleph_0^{\aleph_0} = \aleph_1$ ($\aleph_0$ representing a countable infinite set, $\aleph_1$ the continuum). Thus, the straightforward extension of the method used for finite N leads to a vector space $\tilde{\mathfrak{h}}$ with a continuum of orthonormal *basis* vectors. This is no longer a Hilbert space in the ordinary sense of the word, though the term "non-separable Hilbert space" is used for it in mathematics.

In this connection, we must remember that, for the description of the physical situation, there is no need for such a large space. It is also well known that even for $N = \infty$ there is a representation of (12) within an ordinary Hilbert space. In fact we have used this representation in the previous discussion. It follows from the assumption that there is one vacuum state satisfying (14). Starting from this assumption, the argument can be carried through in essentially the same way as for finite N. The point is, however, that, for infinite N, (14) is no longer a consequence of (12). In other words, there will be different irreducible representations of (12).

One might perhaps be tempted to think that the ambiguity left by the relations (12) or (17) is a matter of mathematical sophistication without relevance to field theory. However, the following examples will show that, starting from the "standard representation" of (12) in $\mathfrak{h}$, we obtain, by the very simplest substitutions, operators in the same space which belong to inequivalent representations. If one disregards the inequivalence and tries to calculate a unitary matrix which connects the two operator systems, one obtains infinite results.

Let us start from operators u_k, $u_k^\dagger$ obeying (12) and (14),

i. e. belonging to the "standard representation" (12). Then, introduce the linear combinations

$$\left.\begin{aligned} v_k &= \cosh\varepsilon\cdot u_k + \sinh\varepsilon\cdot u_k^\dagger, \\ v_k^\dagger &= \sinh\varepsilon\cdot u_k + \cosh\varepsilon\cdot u_k. \end{aligned}\right\} \quad (22)$$

The commutation relations are unchanged. If we take ε to be infinitesimal, we have

$$\left.\begin{aligned} \delta u_k &= v_k - u_k = \varepsilon u_k^\dagger, \\ \delta u_k^\dagger &= v_k^\dagger - u_k^\dagger = \varepsilon u_k. \end{aligned}\right\} \quad (23)$$

Writing this formally as an infinitesimal unitary transformation, the generating operator is

$$T = \frac{i}{2}\Sigma\left(u_k^\dagger u_k^\dagger - u_k u_k\right), \quad (24)$$

i. e.

$$i\varepsilon\,[T u_k] = \varepsilon u_k^\dagger = \delta u_k, \quad i\varepsilon\,[T u_k^\dagger] = \varepsilon u_k = \delta u_k^\dagger.$$

It is easily seen that T is not a proper operator, but transforms every vector of $\mathfrak{h}$ into one with infinite length. However, we can follow the matter a little further by going over to the non-separable space $\tilde{\mathfrak{h}}$ in which a vector is represented by a function $\psi(q_1, q_2 \ldots)$ of infinitely many variables and

$$u_k = \frac{1}{\sqrt{2}}\left(q_k + \frac{\partial}{\partial q_k}\right); \quad u_k^\dagger = \frac{1}{\sqrt{2}}\left(q_k - \frac{\partial}{\partial q_k}\right).$$

One can then work out the effect of the operation $e^{i\varepsilon T}$ and try to recognize its implications for the "physically interesting states" which are a subset of $\tilde{\mathfrak{h}}$, namely the Hilbert space $\mathfrak{h}$ generated by the $u_k^\dagger$ from Φ_0. The result is that $e^{i\varepsilon T}$ transforms every vector of $\mathfrak{h}$ into one which has a zero scalar product with any second vector from $\mathfrak{h}$.

$$(\Phi_2, e^{i\varepsilon T}\Phi_1) = 0 \text{ for every } \Phi_1, \Phi_2 \text{ from } \mathfrak{h}. \quad (25)$$

Nevertheless, $\Phi_1' = e^{i\varepsilon T}\Phi_1$ is a vector with finite norm in $\tilde{\mathfrak{h}}$. In order to understand how this may happen we consider the case of a very large but finite N. Then there is no distinction between $\tilde{\mathfrak{h}}$ and $\mathfrak{h}$. The expressions (25) are slightly different from zero, and of course $\| \Phi_1' \| = \| \Phi_1 \|$. As N becomes larger, the projection of Φ_1' into the subspace belonging to a total "particle number" $\nu = \Sigma \nu_k < n$ (n fixed) becomes smaller and is compensated by a growth of the total probability for $\nu < n$. In this way Φ_1' moves out of $\mathfrak{h}$ in the limit, because there are no states in $\mathfrak{h}$ belonging to an "actually" infinite particle number, though the particle number may be arbitrarily large.

To sum up: $v_k, v_k^\dagger$ are proper operators in the ordinary Hilbert space $\mathfrak{h}$ (according to (22)), obeying the same commutation relations as $u_k, u_k^\dagger$, but there is no proper unitary transformation connecting the two operator systems, i. e. these belong to inequivalent representations of (12). For the representation defined by the $v_k, v_k^\dagger$, there is no "vacuum state" satisfying

$$v_k \Phi_0' = 0.$$

A similar example which is closer to practical calculations in field theory is the following:

We take two free fields which obey the field equations

$$\left.\begin{aligned}(\Box - m_1^2)\, \psi_1(x) &= 0, \\ (\Box - m_2^2)\, \psi_2(x) &= 0,\end{aligned}\right\} \quad (26)$$

and which coincide (including their first time derivatives) for $t = 0$;

$$\left.\begin{aligned}\psi_1(\vec{x}, 0) &= \psi_2(\vec{x}, 0) = \psi(\vec{x}, 0), \\ \dot{\psi}_1(\vec{x}, 0) &= \dot{\psi}_2(\vec{x}, 0) = \pi(\vec{x}, 0),\end{aligned}\right\} \quad (27)$$

and define in the usual way for each field a splitting into a creation and an annihilation part, for instance for ψ_1

$$\left.\begin{aligned}u_1(\vec{p}) &= (2\pi)^{-\frac{3}{2}} \int (E_1 \psi(\vec{x}) + i\pi(\vec{x}))\, e^{i\vec{p}\vec{x}} d\vec{x}, \\ u_1^\dagger(\vec{p}) &= (2\pi)^{-\frac{3}{2}} \int (E_1 \psi(\vec{x}) - i\pi(\vec{x}))\, e^{i\vec{p}\vec{x}} d\vec{x}, \\ E_1 &= \sqrt{\vec{p}^2 + m_1^2}.\end{aligned}\right\} \quad (28)$$

Then the operators $u_1(\vec{p})$, $u_1^\dagger(\vec{p})$ are connected to the $u_2(\vec{p})$, $u_2^\dagger(\vec{p})$ by a transformation like (22), namely

$$u_2(\vec{p}) = \frac{E_1+E_2}{2E_1}\,u_1(\vec{p}) + \frac{E_1-E_2}{2E_1}\,u_1^\dagger(\vec{p}). \tag{29}$$

Now, if there is any state for which

$$u_1(\vec{p}),\ \Phi_0 = 0 \qquad \text{for all } \vec{p},$$

then there is no Φ_0' which satisfies

$$u_2(\vec{p}),\ \Phi_0' = 0 \qquad \text{for all } \vec{p},$$

and vice versa. This may show that the "strange representations" of (12) will almost inevitably turn up in any discussion in field theory.

We have already mentioned that similar considerations apply to Fermi particles. The effect of the Pauli principle is to "reduce" the number of basis vectors in $\tilde{\mathfrak{H}}$ to $2^{\aleph_0}$, which is still the continuum.

The existence of different representations of (12), (13) was discovered some time ago[16], but has not entered into the consciousness of physicists until very recently[17]. A systematic study and classification has been made by WIGHTMAN and GÅRDING[18].

§ 2. Functions of the field operators.

After the basic operators have been defined by commutation relations of the form (18), the conventional field theories proceed to give the Lorentz operators P_μ, $M_{\mu\nu}$ (energy-momentum, angular momentum) as functions of the basic operators. As we have seen, the definition (18) is not complete, but we suppose now that it has been augmented by some auxiliary condition which fixes the representation. Then we meet with a second problem. Almost all simple-looking formal expressions in the $\psi(\vec{x})$, $\pi(\vec{x})$, which we may think of writing down, are actually not proper operators. In most cases, they will even have infinite matrix elements between any two states of $\mathfrak{h}$. What one would like to have, then, is a simple criterion for the class of "sensible functions" of the field operators, allowing to decide immediately whether an expression is acceptable or not. We shall illustrate the problem for the "standard representation" (12) with which we have to deal in the case of the asymptotic fields $\overset{\pm}{\varphi}(x)$. Here, a criterion which satisfies practical purposes can indeed be given easily. The method is well known. Nevertheless, some of the arguments may be recalled. The points I want to emphasize are: 1) the characterization of the class of sensible functions of the operators (18) depends only on the type of representation for these operators, not on their physical meaning; 2) the task is solved for the "standard representation" and may be extended to others as soon as their relation to the "standard representation" is known. This is, for instance, the case for the two examples given in the preceding section.

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[16] J. v. NEUMANN, Composition Math. 6, 1 (1938); K. O. FRIEDRICHS, Math. Aspects of the Quantum Theory of Fields, Interscience Publishers, New York 1953, Chapter on "Myriotic Fields".

[17] VAN HOVE, Physica 18, 145 (1952); WIGHTMAN and SCHWEBER, Phys Rev., in print. I am indebted to Prof. WIGHTMAN for a preprint of this paper.

[18] GÅRDING and WIGHTMAN, Proc. Nat. Acad. Sci. 40, 617 (1954).

Let us start from the equations (16)—(19) and assume (21)[19]. The inversion of (19) is

$$\psi(\vec{x}) = \Sigma q_k f_k(\vec{x}), \quad \pi(\vec{x}) = \Sigma p_k f_k(\vec{x}). \tag{30}$$

Now it is clear that all polynomials in the p_k, q_k (or u_k, $u_k^\dagger$), which involve a finite number of additions and multiplications, are well defined operators. We can apply them, for instance, to any basis vector and obtain again a normalizable state. This is not true for the continuous manifold of operators $\psi(\vec{x})$ which are infinite sums of the u_k, $u_k^\dagger$. In fact one checks easily that $\psi(\vec{x})$ transforms every basis vector $\Phi^{(N)}_{kl\ldots}$ into a vector with infinite length. However, $\psi(\vec{x})$ has at least finite matrix elements, and expressions like $\int f(\vec{x})\,\psi(\vec{x})\,d\vec{x}$ are well defined operators if f is square integrable. Hence, $\psi(\vec{x})$ may be regarded as an improper operator in the same way in which one can regard eigenfunctions in the continuous spectrum as improper state vectors. The situation is worse for $\psi^2(\vec{x})$ which has only infinite matrix elements. The remedy may here be found in the following observation. If we have a power series in the $u_k, u_k^\dagger$ *in which each term is arranged in S-product order*[20] (i. e. all destruction operators stand to the right), then only a finite number of terms contribute to a matrix element between two of the basis vectors $\Phi_{kl\ldots}^{(N)}$. Therefore, any expression which is in S-product order has at least finite matrix elements and may then be regarded in general as an improper operator in a similar way to $\psi(\vec{x})$. The simple way of putting two dots around an expression is a safeguard against infinite matrix elements. If X is supposed to be a proper operator, then $X^\dagger X$ must have finite matrix elements, and vice versa. Thus one has the simple criterion:

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[19] If we identify $\psi(\vec{x})$, $\pi(\vec{x})$ with $\bar{\varphi}(\vec{x})$, $\dot{\bar{\varphi}}(\vec{x})$, respectively, the equations (16) and (19) should be replaced by the somewhat more complicated ones which give the splitting into creation and destruction parts (equ. (28)).

If, in the process of rearranging an expression $X(\psi(\vec{x}), \pi(\vec{x}))$ in S-product order, no explicit infinities occur (i. e. if the contractions are finite), X has finite matrix elements. If the same is true for $X^\dagger X$, then X is a proper operator.

The criterion can be extended to other "discrete" representations[21] as they can be related to the standard representation by substitutions.

III. Applications.

In the following chapters, the specialization to the case mentioned in the introduction, in which we have to deal with only one type of particle, will be made. For convenience, a short description of the formal apparatus is given first.

§ 1. Notations.

The manifold of states of a single spinless particle of mass m is most easily described in momentum space. An arbitrary state is then represented by a function $f(p)$ (p the momentum 4-vector) which needs to be defined only for those values of p corresponding to a possible momentum vector of the particle, i. e. for p lying on the positive-energy shell of the hyperboloid $p^2 + m^2 = 0$. However, it is more convenient to regard $f(p)$ as defined in the whole *positive* cone and reject the irrelevant points of momentum space by a factor $\delta(p^2 + m^2)$ which must appear in all relations of physical significance. Thus the scalar product of two wave functions f_1 and f_2 is defined by

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[20] We use the notations of G. C. Wick, Phys. Rev. 80, 268. The S-product between A and B is indicated by double dots: $:AB:$.

[21] For the definition of discrete representation, see the papers mentioned in ref. 18.

$$(f_2 f_1) = \int f_2^*(p) f_1(p) \delta(p^2 + m^2) dp, \tag{31}$$

where it must always be kept in mind that the integration is essentially one over the positive-energy cone of p only. The transformation properties of the wave functions are

$$D(b) f = f': \qquad f'(p) = e^{ipb} f(p) \quad \text{(translations)}.$$
$$D(\Lambda) f = f'': \qquad f''(p) = f(\Lambda^{-1}, p) \quad \text{(homogeneous L. T.)}.$$

A description in ordinary space coordinates is obtained by the Fourier transformation

$$f(x) = (2\pi)^{-\frac{3}{2}} \int f(p) e^{ipx} \delta(p^2 + m^2) dp. \tag{32}$$

This function satisfies the Klein-Gordon equation

$$(\Box - m^2) f(x) = 0, \tag{33}$$

but contains only positive-energy Fourier components so that the initial condition $f(\vec{x}, 0)$ is sufficient to determine $f(\vec{x}, t)$ for all times. The scalar product in this formulation has the more complicated form

$$(f_2, f_1) = 2i \int f_2^*(\vec{x}, 0) \frac{\partial f_1}{\partial t}(\vec{x}, 0) d\vec{x}. \tag{34}$$

Therefore, $f(\vec{x})$ may not be directly interpreted as the probability amplitude for the position $\vec{x}$.

Let $\bar{u}_k$ be the destruction operator for the state f_k (wave function $f_k(p)$) in the asymptotic configuration at $t = -\infty$ (see Chapter I). Then it is convenient to define the continuous manifold of destruction operators $\bar{u}(p)$ by

$$\bar{u}_k = \int \bar{u}(p)\, f_k^*(p)\, \delta(p^2 + m^2)\, dp, \tag{35 a}$$

or conversely

$$\bar{u}(p) = \Sigma f_k(p)\, \bar{u}_k, \tag{35 b}$$

where it has been assumed that the $f_k(p)$ form a complete system, orthogonal and normalized according to (31). The creation operators $u^\dagger(p)$ are the hermitian adjoints and the relations (35) apply of course equally to the operators $\overset{+}{u}_k$, $\overset{+}{u}(p)$ for the asymptotic configuration at $t = +\infty$.

The commutation relations are most conveniently expressed in the symbolic form

$$\left.\begin{aligned} &[\bar{u}(p)\, \bar{u}^\dagger(p')]\, \delta(p^2 + m^2)\, \delta(p'^2 + m^2) \\ &\qquad = \delta^4(p - p')\, \delta(p^2 + m^2). \end{aligned}\right\} \tag{36}$$

The transformation properties are:

$$\left.\begin{aligned} D(b)\, \bar{u}(p)\, D^\dagger(b) &= e^{-ipb}\, \bar{u}(p) \quad \text{(translation)}, \\ D(\Lambda)\, \bar{u}(p)\, D^\dagger(\Lambda) &= \bar{u}(\Lambda, p) \qquad \text{(homogeneous L. T.)}. \end{aligned}\right\} \tag{37}$$

In analogy to (32) we define

$$\bar{u}(x) = (2\pi)^{-\frac{3}{2}} \int \bar{u}(p)\, e^{ipx}\, \delta(p^2 + m^2)\, dp. \tag{38}$$

The creation and destruction operators are combined to the "incoming field"

$$\bar{\varphi}(x) = \bar{u}(x) + \bar{u}^\dagger(x). \tag{39}$$

§ 2. The existence of general field theories.

As the $\bar{u}(p)$, $\bar{u}^\dagger(p)$ form a complete operator system, an arbitrary operator may be expanded in a series of S-products of the $\bar{u}(p)$, $\bar{u}^\dagger(p)$,

$$X = \Sigma X^{nm}, \tag{40}$$

where X^{nm} contains n creation and m destruction operators,

$$X^{nm} = \int F^{nm}(p';p)\,\bar{u}^{\dagger}(p')\,\bar{u}(p)\,\delta(p'^2+m^2)\,\delta(p^2+m^2)\,dp'\,dp. \tag{40a}$$

Here p' is an abbreviation of the n arguments $p_1' \cdots p_n'$; similarly p stands for $p_1 \cdots p_m$, $\bar{u}(p)$ for the product of the m destruction factors, etc. The integration is extended over the positive-energy cones only. In order that X be invariant under homogeneous L. T., each F^{nm} must be a Lorentz invariant function of its arguments. Invariance of X under translations means that F^{nm} contains a factor $\delta(\Sigma p' - \Sigma p)$. The S-matrix must fulfil both conditions, the operator $\psi(0)$ ("actual field" at the origin) the first.

$$\psi^{nm}(0) = (2\pi)^{-\frac{3}{2}} \int f^{nm}(p';p)\,\bar{u}^{\dagger}(p')\,\bar{u}(p)\,\delta(p'^2+m^2)\,\delta(p^2+m^2)\,dp'\,dp. \tag{41}$$

f^{nm} is an invariant function of n vectors p' and m vectors p, symmetric against any permutation of the variables on either side of the semicolon. Also

$$f^{nm*}(p';p) = f^{mn}(p;p'). \tag{42}$$

From (41) we get $\psi(x)$, replacing f by $fe^{i(\Sigma p - \Sigma p')x}$, and

$$\left.\begin{aligned} \frac{1}{2E_P}\psi^{nm}(\vec{P},t) &\equiv (2\pi)^{-\frac{3}{2}} \int \psi^{nm}(\vec{x},t)\,e^{-i\vec{P}\vec{x}}\,d\vec{x} \\ = \int f^{nm}(p';p)\,\delta(\Sigma\vec{p} - \Sigma\vec{p}' - \vec{P})\,&e^{-i(\Sigma E - \Sigma E')t}\,\bar{u}^{\dagger}(p')\,\bar{u}(p)\,\delta(p'^2+m^2) \\ &\times \delta(p^2+m^2)\,dp'\,dp. \end{aligned}\right\} \tag{43}$$

The letters E mean the energies belonging to the respective momentum vectors, e. g. $E_P = \sqrt{\vec{P}^2 + m^2}$. To fulfil the asymptotic condition (15) we must choose f so that

$$\lim_{t\to-\infty} \psi^{nm}(\vec{P},t) \to 0 \qquad \text{(except for } \psi^{01} \text{ and } \psi^{10}\text{)},$$

$$\lim_{t\to+\infty} \psi^{nm}(\vec{P},t) \neq 0 \qquad \left\{ \begin{array}{l} \text{(for at least some index pairs } n, m \text{ in} \\ \text{order to obtain an } S\text{-matrix different} \\ \text{from the identity).} \end{array} \right.$$

This can only be so if f contains a factor $\delta_+ (\Sigma E - \Sigma E' - \alpha)$ which in turn can only arise from one of the invariant functions

$$h_1 = \delta_+ [(\Sigma p - \Sigma p')^2 + m^2]$$

or

$$h_2 = \delta_- [(\Sigma p - \Sigma p')^2 + m^2].$$

We write

$$f = g_1 h_1 + g_2 h_2. \tag{44}$$

This also gives a separation of ψ into two terms ψ_1 and ψ_2. Taking into account the δ-factor in the spatial momentum components, we can write

$$h_1 = \frac{1}{2\,E_p} [\delta_+ (E_P + \varepsilon) + \delta_+ (E_P - \varepsilon)],$$

where

$$\varepsilon = \Sigma E - \Sigma E'.$$

In the limits $|t| \to \infty$ the factors $he^{-i\varepsilon t}$ will become

	$h_1 e^{-i\varepsilon t}$	$h_2 e^{-i\varepsilon t}$
$t \to -\infty$	$\frac{1}{2\,E_P} e^{iE_P t} \delta (E_P + \varepsilon)$	$\frac{1}{2\,E_P} e^{-iE_P t} \delta (E_P - \varepsilon)$
$t \to +\infty$	$\frac{1}{2\,E_P} e^{-iE_P t} \delta (E_P - \varepsilon)$	$\frac{1}{2\,E_P} e^{iE_P t} \delta (E_P + \varepsilon)$

To satisfy the asymptotic condition at $t = -\infty$ the function g_1 must vanish in a region around $\varepsilon = -E_P$ or, in other words, the variability domain of the 4-vector

$$q = \Sigma p - \Sigma p' \tag{45}$$

must be so restricted by g_1 that in the backward cone values giving $q^2 + m^2 = 0$ are excluded. Similarly, g_2 must exclude values $q^2 + m^2 = 0$ in the forward cone. If this is satisfied, then

$$\left.\begin{aligned} \overset{+}{u} (P) &= \sum_{n,m} \int g_1^{nm} (p'; p)\, \bar{u}^\dagger (p')\, \bar{u} (p)\, \delta (\Sigma p - \Sigma p' - P) \\ &\qquad \delta (p'^2 + m^2)\, \delta (p^2 + m^2)\, dp'\, dp \\ \overset{+}{u}{}^\dagger (P) &= \sum_{n,m} \int g_2^{nm} (p'; p)\, \bar{u}^\dagger (p')\, \bar{u} (p)\, \delta (\Sigma p - \Sigma p' + P) \\ &\qquad \delta (p'^2 + m^2)\, \delta (p^2 + m^2)\, dp'\, dp. \end{aligned}\right\} \tag{46}$$

As g_1 and g_2 are arbitrary for the relevant values of their arguments it may be expected that the frame is wide enough to give any S-matrix. Indeed this can be checked easily in the limit of very weak interaction.

§ 3. Commutation relations of the field variables.

The representation of the field as

$$\left.\begin{aligned} \psi(x) = (2\pi)^{-\frac{3}{2}} \Sigma \int f^{nm}(p'; p)\, e^{i(\Sigma p - \Sigma p')x}\, \bar{u}^{\dagger}(p')\, \bar{u}(p) \\ \delta(p'^2 + m^2)\, \delta(p^2 + m^2)\, dp'\, dp \end{aligned}\right\} \quad (47)$$

and the knowledge we have obtained about the structure of the f^{nm} allow a few statements about the commutation relations. If we form the product $\psi(x')\,\psi(x)$ and rearrange it in S-product order we get

$$\left.\begin{aligned} \psi(x')\,\psi(x) = (2\pi)^{-3}\, \Sigma s! \binom{m'}{s} \binom{n}{s} \int d\bar{p}'\, d\bar{p}\, dp'\, dp \\ \overbrace{\bar{u}^{\dagger}(\bar{p}')}^{n'}\, \overbrace{\bar{u}^{\dagger}(p')}^{n-s}\, \overbrace{\bar{u}(\bar{p})}^{m'-s}\, \overbrace{\bar{u}(p)}^{m} \times e^{i(\Sigma\bar{p} - \Sigma\bar{p}')x' + i(\Sigma p - \Sigma p')x} \\ \int f^{n'm'}(\bar{p}'; \bar{p}, q)\, f^{nm}(p', q; p)\, e^{i\Sigma q(x'-x)}\, \delta(\varkappa^2)\, dq, \end{aligned}\right\} \quad (48)$$

where $\delta(\varkappa^2)$ stands for the product of the $\delta(p^2 + m^2)$ for all the momentum vectors which appear. There are s momentum vectors q in the last integral. The commutator $[\psi(x')\,\psi(x)]$ is then obtained by subtracting the same term with x and x' interchanged. By virtue of the asymptotic condition we must have

$$f^{10}(q) = f^{01}(q) = 1. \quad (49)$$

We first look at the *vacuum expectation value* of (48). Here only the terms with $m = 0$; $m' = s = n$; $n' = 0$ contribute.

$$\left.\begin{aligned} \langle 0 \,|\, \psi(x')\,\psi(x) \,|\, 0 \rangle \\ = (2\pi)^{-3}\, \Sigma s! \int |\, f^{s0}(q)\,|^2\, e^{i(\Sigma q)(x'-x)}\, \delta(q^2 + m^2)\, dq. \end{aligned}\right\} \quad (50)$$

Because of the relations

$$i\Delta^{+}(\xi; a) = (2\pi)^{-3} \int_{Q_0 > 0} e^{iQ\xi}\, \delta(Q^2 + a^2)\, dQ, \quad (51)$$

$$\Delta^{+}(\xi; a) - \Delta^{+}(-\xi; a) = \Delta(\xi; a) \tag{52}$$

we can write

$$\left.\begin{aligned} & -i \langle 0 \,|\, [\psi(x')\, \psi(x)] \,|\, 0 \rangle \\ & = \Delta(x' - x; m) + \sum_{s=2}^{\infty} \int F^{(s)}(a)\, \Delta(x' - x; a)\, da^2 \end{aligned}\right\} \tag{53}$$

with

$$F^{(s)}\left(\sqrt{-Q^2}\right) = s! \int |f^{s0}(q)|^2\, \delta(\Sigma q - Q)\, \delta(q^2 + m^2)\, dq. \tag{54}$$

For equal times ($t' = t$)

$$\left.\begin{aligned} & \langle 0 \,|\, [\psi(x')\, \psi(x)] \,|\, 0 \rangle = 0 \\ & \langle 0 \,|\, [\dot{\psi}(x')\, \psi(x)] \,|\, 0 \rangle = -i\delta(\vec{x} - \vec{x}') \left[1 + \sum_{2}^{\infty} \int F^{(s)}(a)\, da^2\right] \\ & \qquad = -i\,\delta(\vec{x} - \vec{x}') \,.\, C. \end{aligned}\right\} \tag{55}$$

$F^{(s)}(a)$ is a positive, real function and hence $C > 1$, unless all f^{n0} vanish for $n \neq 1$.[22]

In a conventional theory (in which $[\psi(x)\, \psi(x')] = 0$ for equal times) the latter alternative may be excluded. One can, for instance, calculate the matrix element of the commutator $[\psi(x')\, \psi(x)]$ between the vacuum and a two-particle state. Under the assumption $f^{n0} = 0$ for $n \neq 1$, this gives

$$\begin{aligned} & \langle p_1, p_2 \,|\, [\psi(x')\, \psi(x)] \,|\, 0 \rangle \\ & = e^{-i(p_1 + p_2)x'} \int f^{21}(p_1, p_2; q)\, e^{iq(x' - x)}\, \delta(q^2 + m^2)\, dq \\ & - e^{-i(p_1 + p_2)x} \int f^{21}(p_1, p_2; q)\, e^{-iq(x' - x)}\, \delta(q^2 + m^2)\, dq. \end{aligned}$$

This expression should then vanish for arbitrary p_1, p_2, $\vec{x}'$ and $\vec{x}$ as soon as $t' = t$. This can only be true if f^{21} is identically zero and this, in turn, would mean that there is no elastic scattering.

Thus, in a conventional theory, it is necessary that f^{n0} differs from zero at least for some value of $n \neq 1$. According to (55), this then implies that $\dot{\psi}(x)$ cannot be identified with the canonically conjugate momentum of the field, though one could perhaps have[23]

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[22] The formulae (53) — (55) have been given previously by H. LEHMANN, Nuovo Cimento 11, 342 (1954). The derivation here is essentially LEHMANN's.

$$\dot{\psi}(\vec{x}) = C\,\pi(\vec{x}) \quad \text{with} \quad C > 1. \tag{56}$$

§ 4. Conventional theory.

We suppose the theory to be defined in terms of operators $\psi(\vec{x}), \pi(\vec{x})$ obeying (18). The commutation relations are the same as those for the $\overline{\varphi}(\vec{x}), \dot{\overline{\varphi}}(\vec{x})$ and it appears therefore natural to ask whether a unitary operator R exists which transforms the one set into the other.

$$\psi(\vec{x}) = R\,\overline{\varphi}(\vec{x})\,R^{\dagger}, \quad \pi(\vec{x}) = R\,\dot{\overline{\varphi}}(\vec{x})\,R^{\dagger}. \tag{57}$$

In other words: can $\psi(\vec{x}), \pi(\vec{x})$ belong to the same representation of the canonical commutator ring as $\overline{\varphi}(\vec{x}), \dot{\overline{\varphi}}(\vec{x})$? This is usually assumed and, in fact, $R^{\dagger}$ is Dyson's matrix $U(0;-\infty)$.

The canonical momenta $\pi(\vec{x})$ need not have simple relativistic transformation properties. But, with respect to translations in space, we shall also require

$$D(\vec{b})\,\pi(\vec{x})\,D^{\dagger}(\vec{b}) = \pi(\vec{x}-\vec{b}). \tag{58}$$

(58) holds in all customary theories (for instance if $\pi(\vec{x})$ is proportional to $\dot{\psi}(\vec{x})$) and, indeed, any other assumption but (58) would appear extremely unnatural. From (58) and the analogous equation for $\psi(\vec{x})$ it follows that R must commute with the space translations

$$[R\,\vec{P}] = 0. \tag{59}$$

This is also a well known fact in conventional theories (conservation of spatial momentum in all virtual processes). Now the spectrum of $\vec{P}$ is continuous, apart from a single discrete eigenvalue 0 which belongs to the vacuum state Φ_0. If we apply (59) to Φ_0 we obtain

$$\vec{P}R\,\Phi_0 = \vec{P}\Phi_0' = 0. \tag{60}$$

If R is a unitary operator, then Φ_0' is again a normalized state and (60) indicates that

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[23] If one drops the factor C in (55), (56), one has to take it up in the asymptotic condition. This is only a different way of expressing the same thing. Cf. ref. 21 and G. Källén, Helv. Phys. Acta 25, 417 (1952).

$$\Phi_0' = \Phi_0, \tag{61}$$

because there are no other discrete eigenstates of $\vec{P}$. In all theories considered so far, (61) is contradicted immediately by the form of the Hamiltonian. However, we can also disprove (61) without reference to any particular form for H. This equation would imply

$$v(\vec{P})\,\Phi_0 = 0 \qquad \text{for all } \vec{P},$$

with

$$v(\vec{P}) = R\,\pi(\vec{P})\,R^{\dagger} = E_P\,\tilde{\psi}(\vec{P}) + i\tilde{\pi}(\vec{P}).$$

Here $\tilde{\psi}(\vec{P})$, $\tilde{\pi}(\vec{P})$ are the Fourier transforms of $\psi(\vec{x})$ and $\pi(\vec{x})$. Therefore

$$\langle a\,|\,\pi(\vec{P})\,|\,0\rangle = iE_P\langle a\,|\,\tilde{\psi}(\vec{P})\,|\,0\rangle,$$

$$\langle 0\,|\,\pi(\vec{P})\,|\,a\rangle = -iE_P\langle 0\,|\,\tilde{\psi}(\vec{P})\,|\,a\rangle$$

and, thus,

$$\left.\begin{aligned} &\langle 0\,|\,[\pi(\vec{P}')\,\psi(\vec{P})]\,|\,0\rangle \\ &= -iE'\langle 0\,|\,\tilde{\psi}(\vec{P}')\,\tilde{\psi}(\vec{P}) + \tilde{\psi}(\vec{P})\,\tilde{\psi}(\vec{P}')\,|\,0\rangle \\ &= \delta(\vec{P}'-\vec{P})\left(1+\sum_{s=2}^{\infty}\int\frac{\sqrt{\vec{P}^2+m^2}}{\sqrt{\vec{P}^2+a^2}}F^{(s)}(a)\,da^2\right) = C'\,\delta(P'-P) \end{aligned}\right\} \tag{62}$$

with $C' > 1$ which contradicts the commutation relations. We conclude:

The unitary matrix R of (57) *cannot exist, and the same applies to the "free field vacuum" of the Tamm—Dancoff method.*[24]

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[24] The question may be raised as to whether the non-existence of Φ_0' represents a serious obstacle against the use of the Tamm-Dancoff method in practical calulations. One may argue that, loosely speaking, Φ_0' is a state which has zero expansion coefficients with respect to any orthogonal system of physically interesting states. Nevertheless, the *ratio* of these coefficients is finite and mathematically definable; for instance, we can regard Φ_0' as an eigenfunction of the momentum belonging to the continuous spectrum. It may be hoped then that the normalization factor zero will not enter into the final expressions for physical quantities. Now it appears probable that this is indeed true as long as we deal with collision problems in which Φ_0' is only used in an intermediate stage of the calculations and we get back to the physical vacuum by passing to the limits $|t| \to \infty$. On the other hand, if the method is used for the determination. of bound states, the situation is worse, because for these problems the difference between continuous and discrete spectrum is essential and the relation of the final result to the basis system built up from Φ_0' cannot be eliminated. The failure to renormalize the Tamm-Dancoff method in these cases is probably intimately connected to the non-existence of Φ_0'.

Thus, if a scheme on the lines of (18) is possible at all, one must be careful about the type of representation for the operators. The "obvious" choice (57) is excluded.

§ 5. Study of the commutation relations by perturbation calculation.

Do fields exist which satisfy

$$[\psi(x')\,\psi(x)] = 0 \qquad \text{for } (x'-x)^2 > 0\,? \tag{63}$$

This is a much more general question since we leave it open how the conjugate momenta $\pi(\vec{x})$ may be defined and what type of representation is meant. It can be studied by going back to (48). (63) gives some integral equations for the f^{nm} and the question is whether these have solutions. Of course, a rigorous discussion appears hopeless, but a perturbation calculation is possible. We put

$$f^{nm} = {}^{(0)}f^{nm} + \varepsilon\;{}^{(1)}f^{nm} + \varepsilon^2\;{}^{(2)}f^{nm} + \cdots \tag{64}$$

with the zero approximation

$$\left.\begin{aligned} &{}^{(0)}f^{nm} = 0 \qquad \text{for } n, m \neq 0{,}1 \text{ or } 1{,}0, \\ &{}^{(0)}f^{01} = {}^{(0)}f^{10} = 1, \end{aligned}\right\} \tag{65}$$

corresponding to ${}^{(0)}\psi(x) = \overline{\varphi}(x)$. Then we obtain in first order, setting $x' = 0$,

$$\int \delta(q^2+m^2)\,dq\,\big[(l+1)\,{}^{(1)}f^{k,l+1}(p',p;q)\,(e^{-ipx} - e^{i(\Sigma p-\Sigma p'+q)x}) + (k+1)^{(1)}f^{k+1,l}(p',q;p)\,(e^{i(\Sigma p-\Sigma p'-q)x} - e^{iqx}) = 0$$

for $x^2 > 0$.
For $t = 0$ the equation must hold for all $\vec{x}$. Taking the three-dimensional Fourier transform, we have the following functional equation for the f:

$$\left.\begin{aligned} &\frac{1}{2E_q}\big[(l+1)\,{}^{(1)}f^{k,l+1}(p';p,q) - (k+1)\,{}^{(1)}f^{k+1,l}(p',\overline{q};p)\big] \\ = &\frac{1}{2E_P}\big[(l+1)\,{}^{(1)}f^{k,l+1}(p';p,P) - (k+1)\,{}^{(1)}f^{k+1,l}(p',\overline{P};p). \end{aligned}\right\} \tag{66}$$

Here $\overline{q}$ stands for the four-vector $(-\vec{q} + E_q)$; similarly $\overline{P} = (-\vec{P} + E)$ and $\vec{P} = \Sigma \vec{p}' - \Sigma \vec{p} - \vec{q}$.

The f-functions are Lorentz invariant functions if their arguments are regarded as 4-vectors with square length m^2. It is seen that by (66) only those f^{nm}-functions are coupled which belong to the same value of $\nu = n + m$ so that each value of ν may be treated independently. We discuss the first non-trivial case ($\nu = 2$). Here, because of the Lorentz invariance, we are dealing with functions of only one argument. It is convenient to put

$$^{(1)}f^{11}(p'; p) = F\left[-(p' - p)^2 - m^2\right],$$

$$2\,^{(1)}f^{20}(p_1', p_2') = G\left[-(p_1' + p_2')^2 - m^2\right].$$

Then (66) with $k = 1$, $l = 0$ gives a functional equation for $F(x)$ and $G(x)$ which may be solved. One finds the general solution[25]

$$F(x) = G(x) = \frac{A}{x} + B + Cx. \tag{67}$$

The case $\nu = 2$ which we have considered corresponds to an interaction Hamiltonian which involves (at least in the first order) only terms containing three factors of the field variables (three-coupling). Thus (67) says that, in first approximation, there are only three types of three-coupling, compatible with our general assumptions. The A-coupling gives rise to the field equation

$$(\Box - m^2)\,\psi(x) = \frac{A}{2}\,\psi^2(x) \tag{68}$$

and the interaction Hamiltonian

$$V = \int \psi^3(\vec{x})\, d\vec{x}.$$

B and C produce derivative couplings. We single out the case A for further discussion. Taking advantage of the x-representation in which the formulae are more compact, we can write

$$^{(1)}\psi(x) = -\int \Delta_R(x, 1)\colon \varphi^2(1)\colon dx_1. \tag{69}$$

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[25] The calculation is given in the CERN manuscript T/RH-1, Copenhagen, March 1954.

The extension of the argument to the second order is straightforward. One finds

$$ {}^{(2)}\psi(x) = 2\int \varDelta_R(x,1)\,\varDelta_R(1,2)\colon \varphi^2(2)\,\varphi(1)\colon dx_1\,dx_2. \tag{70} $$

This agrees with the formal solution of (68) by the method of YANG-FELDMAN and KÄLLÉN apart from the fact that there we would obtain the additional term

$$ \int \varDelta_R(x,1)\,\varDelta_R(1,2)\,\varDelta^{(1)}(1,2)\,\varphi(2)\,dx_1\,dx_2. $$

Now, because of (38), (39), an operator of the form $\int f(x-x_1)\,\varphi(x_1)\,dx_1$ can only be a multiple of $\varphi(x)$ (if it is supposed to have well defined matrix elements). Such a term would be without physical interest and is furthermore excluded here by the asymptotic condition.

In the third order we meet with the well known difficulties from products of singular functions. It is interesting to note that, in a perturbation calculation, this is an inevitable consequence of (63), irrespective of the definition of the conjugate momenta or of the form of the Hamilton function. The mathematical reason is that (63) implies in the lowest order a "local" interaction, i. e. an expression for ${}^{(1)}\psi(x)$ involving products of $\varphi(x')$ at the same point. In the higher orders we obtain the contractions of these powers of $\varphi(x')$, that is to say products of $\varDelta$-functions of the "closed loop" type. It is not the object of this paper to discuss whether these difficulties are really serious or whether they could be overcome in a satisfactory manner by a careful definition of the limiting processes. We shall here merely indicate the extension of (69), (70) to the third order.

The troublesome terms are those which involve two contractions in the commutator $[{}^{(2)}\psi(x)\,{}^{(1)}\psi(x)]$. They are

$$ \alpha = \int \varDelta_R(\overline{x},1)\,\varDelta_R(x,3)\,\varDelta_R(1,2)\,\varDelta(2,3)\,\varDelta^{(1)}(2,3)\,\varphi(1)\,dx_1\,dx_2\,dx_3, $$

$$ \beta = \int \varDelta_R(\overline{x},1)\,\varDelta_R(x,3)\,\varDelta_R(1,2)\,\varDelta(1,3)\,\varDelta^{(1)}(2,3)\,\varphi(2)\,dx_1\,dx_2\,dx_3, $$

$$ \gamma = \int \varDelta_R(\overline{x},1)\,\varDelta_R(x,3)\,\varDelta_R(1,2)\,\varDelta(2,3)\,\varDelta^{(1)}(1,3)\,\varphi(2)\,dx_1\,dx_2\,dx_3. $$

There is some ambiguity in these expressions, most evident in the case of α which contains $\Delta(\xi)\,\Delta^{(1)}(\xi)$. A calculation in momentum space with a suitable order of integration gives the convergent result

$$\alpha \sim \int \Delta_R(\overline{x} - x_1) \int_{2m}^{\infty} da \frac{\sqrt{a^2 - 4m^2}}{(a^2 - m^2)^2} \Delta(x_1 - x;\, a)\, \varphi(x_1)\, dx_1 .$$

If we put

$$F(x - x_1) = \int_{2m}^{\infty} da \frac{\sqrt{a^2 - 4m^2}}{(a^2 - m^2)^2} \Delta_R(x - x_1;\, a),$$

then we can write for equal times of $\overline{x}$ and x

$$\alpha \sim \int F(x - x_1)\, \Delta(\overline{x} - x_1)\, \varphi(x_1)\, dx_1 ,$$

and this may be compensated by a term from $[{}^{(3)}\psi(x)\, {}^{(0)}\psi(\overline{x})]$ if we put

$${}^{(3)}\psi'(x) \sim \int F(x - x_1) : \psi^2(x_1) : dx_1 .$$

The same technique of shifting the retardation sign from one of the functions $\Delta_R(\overline{x}, 1)$, $\Delta_R(x, 3)$ may be applied in the case β and γ. If we add to β and γ the antisymmetric supplement arising from $[{}^{(2)}\psi(x)\, {}^{(1)}\psi(\overline{x})]$, we have

$$\int \Delta_R(\overline{x}, 1)\, \Delta_R(x, 3)\, \varphi(2) \big\{ [\Delta_R(1, 2)\, \Delta(1, 3)\, \Delta^{(1)}(2, 3)$$
$$- \Delta_R(3, 2)\, \Delta(3, 1)\, \Delta^{(1)}(2, 1)] + [\Delta_R(1, 2)\, \Delta(2, 3)\, \Delta^{(1)}(1, 3)$$
$$- \Delta_R(3, 2)\, \Delta(2, 1)\, \Delta^{(1)}(1, 3)] \big\}\, dx_1\, dx_2\, dx_3 .$$

The two square brackets we call K_1 and K_2, respectively, and

$$\int K_1\, \varphi(2)\, dx_2 = \int \Phi_1(x_1, x_3, P)\, \varphi(P)\, \delta(P^2 + m^2)\, dP,$$

$$\int K_2\, \varphi(2)\, dx_2 = \int \Phi_2(x_1, x_3, P)\, \varphi(P)\, \delta(P^2 + m^2)\, dP.$$

Similar to α, Φ_1 and Φ_2 may be defined as convergent expressions which vanish for space-like $(x_1 - x_3)$ and which are antisymmetrical in x_1, x_3. We can split them like $\Delta(x_1 - x_3)$

into a retarded and an advanced part. For equal times of $\bar{x}$ and x we can use

$$\Delta_R(\bar{x}, 1)\,\Delta_R(x, 3)\,\Phi_R(1, 3) = \Delta_R(\bar{x}, 1)\,\Delta(x, 3)\,\Phi_R(1, 3),$$

$$\Delta_R(\bar{x}, 1)\,\Delta_R(x, 3)\,\Phi_A(1, 3) = \Delta(\bar{x}, 1)\,\Delta_R(x, 3)\,\Phi_A(1, 3).$$

Such expressions may be compensated by a suitable term in ${}^{(3)}\psi$.

These remarks seem to indicate that it is possible to carry on the expansion (69), (70) to higher orders, but a more thorough investigation of this point is necessary.

Summary.

An investigation is made of the possibility of defining a theory which is in accord with the principles of quantum physics and special relativity and which describes the interaction processes of particles. There is no contradiction between these three requirements, and a simple mathematical expression for the combination of them is given in equation (1). One may then regard field theory as an extension of quantum mechanics to a system with infinitely many degrees of freedom in such a way that (1) is satisfied. The fact that we are dealing with infinitely many degrees of freedom gives rise to some mathematical problems which—though not generally recognized—have been solved to a large extent. They involve the ambiguity of the canonical commutation relations

$$[p_k, q_l] = -i\delta_{kl}; \qquad k, l = 1, 2 \cdots \infty$$

and the question how to recognize whether some function $F(p, q)$ is a proper operator, an improper operator or a senseless expression.

In comparison with the general set up of a relativistic quantum theory of particles the conventional field theories introduce some additional requirements. The basic variables are there supposed to have 1) the simple relativistic transformation properties of a field, 2) vanishing commutators for equal times, and 3) they

should satisfy the asymptotic condition (15). Taken together these requirements imply a very strong restriction and it is not clear whether they are not actually incompatible if equ. (1) is taken into account[26]. We have proved that if 2) is dropped we have again the wide class of possible theories allowed by the general considerations of Chapter I. The same holds of course if both 1) and 3) are dropped. It does not seem to help, however, to leave out only 1) as long as we still want relativistic invariance of the S-matrix.

In the lowest orders of a perturbation expansion the assumptions are compatible and have physical significance as borne out by the experience from quantum electrodynamics and β-decay.

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[2]· Of course the trivial solution of uncoupled fields is always possible.

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A THEOREM ON INVARIANT ANALYTIC FUNCTIONS WITH APPLICATIONS TO RELATIVISTIC QUANTUM FIELD THEORY

BY

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Introduction.

In a preceding paper[1], the second-named author showed that the main content of a relativistic quantum theory of a scalar field, $\varphi(x)$, is contained in the vacuum expectation values, $F^{(n)}$, defined by

$$F^{(n)}(x_1, \ldots x_n) = (\Psi_0, \varphi(x_1) \ldots \varphi(x_n)\Psi_0),\ n = 1, 2, \ldots,$$

where Ψ_0 is the vacuum state. It was shown there that, as a consequence of the transformation law of the field under space-time translations and the absence of negative energy states, the distributions $F^{(n)}$ are boundary values of analytic functions. The analysis of the structure of the $F^{(n)}$ was carried further, using a theorem, quoted there without proof, which may be stated roughly as follows: an analytic function of n four-vector variables invariant under the orthrochronous Lorentz group is an analytic function of their scalar products.

The first part of the present paper is devoted to a proof of this theorem. Because the techniques introduced in the proof have further useful applications in quantum field theory, we have given a detailed exposition.

The second part of the paper contains a preliminary study of the set, $\mathfrak{M}_n$, of symmetric $n \times n$ complex matrices, Z, defined by $Z_{jk} = z_j \cdot z_k$, $j, k = 1, \ldots n$, where $z_1, \ldots z_n$ are complex four vectors of the form $z_j = \xi_j - i\eta_j$, with ξ_j and η_j real and η_j in the interior of the future light cone (this set of $z_1, \ldots z_n$ is called the tube). According to the theorem proved in the first part of the paper, the set $\mathfrak{M}_n$ is a domain of analyticity of the invariant analytic function which has the physical $F^{(n)}$ as its boundary value when all $\eta_j \rightarrow 0$. It is shown that there are points, $z_1, \ldots z_n$, on the boundary of the tube which yield matrices, $z_i \cdot z_j$, of scalar products lying in the interior of $\mathfrak{M}_n$. From this simple geometrical fact, it follows that an invariant function analytic in the tube cannot have an arbitrary invariant distribution as boundary value. In fact, it turns out that the boundary value has to be an analytic function of the real variables $\xi_j \cdot \xi_k$, $j, k = 1, \ldots n$ in a certain domain and that the analytic function is uniquely determined once its values are known in certain subdomains of the boundary of the tube.

In the third part of the paper, the results of the preceding sections are applied to the vacuum expectation values of a scalar field. It is shown that $F^{(n)}(x_1, \ldots x_n)$ is an analytic function of the real variables $(x_j - x_{j+1}) \cdot (x_k - x_{k+1}), j, k = 1, \ldots n-1$ when all $x_j - x_{j+1}$ are space-like and lie in a certain region. It is further shown that the values of $F^{(n)}$ for all values of its arguments are uniquely determined in terms of its values for space-like separation. For the cases $n = 2, 3, 4$, an even stronger result is obtained: $F^{(n)}$ is determined everywhere from its values at points where all the times $(x_{j0}), j = 1. \ldots n$ are equal. These results are applied to prove generalizations of two theorems of R. Haag, which can be stated roughly as follows: First, let there be given two theories of a field which transforms as a scalar under the rotations and translations of three space at a fixed time. Suppose that the canonical variables of the theories are unitary equivalent at that time via a unitary transformation V. Then, the representations of the Euclidean group of the two theories are unitary equivalent via V. Second, if the two theories just described are, in addition, invariant under the inhomogeneous Lorentz group and have no negative energy states and unique vacuum states, then the vacuum expectation values $(\Psi_0, \varphi(x_1) \ldots \varphi(x_n) \Psi_0)$ are identical in the two theories for $n = 1, 2, 3, 4$. The paper closes with a discussion of the physical significance of this generalized Haag's theorem.

We want to emphasize that the main results of the paper, as far as the structure of the $F^{(n)}$ are concerned (its determination everywhere in terms of its values for space-like separated arguments), are valid in both local and non-local field theory.

1. An Invariant Analytic Function of Vectors is an Analytic Function of Scalar Products.

The following theorem was stated without proof in I.

Theorem 1.

Let f be a complex valued function of n four-vector variables, $z_j = \xi_j - i\eta_j$, $j = 1, \dots n$, where ξ_j and η_j are real. Suppose f is analytic in the tube defined by

$$\infty < \xi_{j\mu} < \infty, \; j = 1, \dots n, \; \mu = 0, 1, 2, 3;$$

$$\eta_j \text{ in the future cone, i. e. } \eta_j^2 > 0, \eta_{j0} > 0, \; j = 1, \dots n$$

and invariant under the orthochronous homogeneous Lorentz group, $L^\uparrow$:

$$f(z_1, \dots z_n) = f(\Lambda z_1, \dots \Lambda z_n) \text{ for } \Lambda \,\varepsilon\, L^\uparrow. \tag{1}$$

Then, f is a function of the scalar products $z_j \cdot z_k$, $j, k = 1, \dots n$. It is analytic on the complex variety, $\mathfrak{M}_n$, over which the scalar products vary when the vectors $z_1, \dots z_n$ vary over the tube.

Outline of the Proof.

If a function is analytic in the tube and satisfies (1) for $\Lambda \,\varepsilon\, L^\uparrow$, then (Lemma 1) it also satisfies (1) when Λ is an arbitrary complex Lorentz transformation, i. e., a matrix $\Lambda^\mu{}_\nu$ $\mu,\nu = 0, 1, 2, 3$, whose elements are complex numbers satisfying $\Lambda^\mathsf{T}\Lambda = 1$, which means

$$\sum_{\mu=0}^{3} \Lambda_\mu{}^\varkappa \Lambda^\mu{}_{\nu'} = g^\varkappa{}_\nu . \tag{2}$$

We call the set of such matrices the *complex Lorentz group*, $\mathfrak{L}$. Furthermore, if $\Lambda z_1, \dots \Lambda z_n$ lies outside the future tube, (1) defines a single valued analytic continuation of the function f originally given. We shall refer to the set of points $\Lambda z_1, \dots \Lambda z_n$, for $\Lambda \,\varepsilon\, \mathfrak{L}$ and $z_1, \dots z_n$ in the tube, as the *extended tube.*

Next, (Lemma 2), we examine the sets of n vectors $z_1, \dots z_n$ and $\zeta_1, \dots \zeta_n$ which satisfy

$$z_j \cdot z_k = \zeta_j \cdot \zeta_k \quad j, k = 1 \dots n. \tag{3}$$

If the $n \times n$ matrix of the scalar products has rank three or four, there exists a complex Lorentz transformation Λ such that

$$\Lambda z_i = \zeta_i \quad i = 1, \dots n.$$

If the rank is two or one, the connection between z_i and ζ_i is more complicated:

$$\Lambda z_i = \zeta_i + \alpha_i \omega,$$

where the α_i, $i = 1, \dots n$ are complex numbers and ω is a vector of zero length orthogonal to Λz_i and ζ_i, $i = 1, \dots n$. For points at which the rank is three or four, the invariance of an analytic function, f, of n vectors $z_1, \dots z_n$, clearly implies the single valuedness of f regarded as a function of the scalar products (3). For points where the rank is two or one, a further argument is necessary and is supplied. Thus, f is a single valued function everywhere on the variety, $\mathfrak{M}_n$ defined by the scalar products $z_j \cdot z_k$, $j, k = 1, \dots n$. The points of $\mathfrak{M}_n$ are labeled by the $1/2\ n(n+1)$ scalar products. $\mathfrak{M}_n$ can be regarded as an algebraic variety in the space of all complex $n \times n$ symmetric matrices. In fact, it turns out to be an open subset of the set of all complex symmetric matrices of rank $\leqslant 4$ and, as such, has dimension $1\ (n=1)$, $3\ (n=2)$, $4n-6\ (n \geqslant 3)$.

In order to be able to connect the continuity properties of invariant functions of vectors with their corresponding properties regarded as functions of scalar products, it is necessary to investigate the connection between neighbourhoods of sets of vectors and neighbourhoods of their sets of scalar products (Lemma 3). This connection is quite a simple one at points of $\mathfrak{M}_n$ where the rank is three or four, but where it is two or one the situation is quite delicate, because the structure of the set of points in the space of the vectors which map into a given point of $\mathfrak{M}_n$ is essentially more complicated. Nevertheless, the proof of the continuity of f as a function on $\mathfrak{M}_n$ can be, and is, carried out.

To complete the proof of the theorem, it remains to show that f is analytic on $\mathfrak{M}_n$. For $n \leqslant 4$, analyticity is a perfectly straightforward notion because $\mathfrak{M}_n$ is an open set in complex Euclidean $1/2\ n(n+1)$ space. However, for $n \geqslant 5$, $\mathfrak{M}_n$ is an open set on a $4n-6$ dimensional algebraic variety and the notion of analyticity requires some explanation. For a point P of $\mathfrak{M}_n (\geqslant 5)$ for which the rank is four, the tangent space to $\mathfrak{M}_n$ at P has dimension $4\,n-6$. (Recall that the tangent space to $\mathfrak{M}_n$ at P is the linear manifold of the space of all complex symmetric matrices spanned by the tangent vectors to $\mathfrak{M}_n$ at P.) Sufficiently small neighbourhoods of P on $\mathfrak{M}_n$ can be put in analytic one to one correspondence with sufficiently small neighbourhoods in the tangent space. Near such points, f can be regarded as defined in a neighbourhood of the origin in a complex $(4\,n-6)$-dimensional Euclidean space and its analyticity defined in the well-known way. The points of $\mathfrak{M}_n$, $n \geqslant 5$, where the rank is less than four are *singular* in the terminology of algebraic geometry 2*. For them, the tangent vector space has dimension $1/2\; n(n+1)$ and neighbourhoods are not locally Euclidean. (The reader may find it helpful to think of the example of the light cone. In that case, the point where the tips of the past and future cones touch is singular and its neighbourhoods are not locally Euclidean. However, it should be borne in mind that the actual situation is much more complicated since singular points only appear on $\mathfrak{M}_n$ for $n \geqslant 5$ and in the simplest case, $n=5$, already form a variety of 24 (real) dimensions.) Evidently, the above definition of analyticity does not apply at such a point. It is not impossible to extend the notion of analyticity to apply there. In fact, one can do it in a number of different ways. However, it can happen that physically important consequences of ordinary analyticity do not hold for "generalized analyticity". In the following, we prove analyticity at all points of $\mathfrak{M}_n$ for $n \leqslant 4$, analyticity at non-singular points for $n \geqslant 5$, and boundedness and continuity at singular points for $n \geqslant 5$, and that is what is to be understood by "analytic on $\mathfrak{M}_n$" in the statement of the theorem. It actually is sufficient to guarantee analyticity in the sense of BOCHNER and MARTIN[3].

The proof of the analyticity is completed in four steps. First, differential equations are derived which f satisfies by virtue of its invariance under $L^{\uparrow}$ or $\mathfrak{L}$ (Lemma 4). The scalar products $z_i \cdot z_j$, $j=1, \ldots n$, considered as functions of the vectors $z_1, \ldots z_n$, satisfy these differential equations. Next, it is shown that, in a neighbourhood of a point of $\mathfrak{M}_n$ at which the scalar products span all solutions of the differential equations, f is expandable in a power series of appropriately chosen scalar products (Lemma 5). Lemmas 6 and 7 then show that at every non-exceptional point the scalar products satisfy the conditions of Lemma 5. Finally, to complete the proof, a theorem on removable singularities is used to show that f is analytic even at the exceptional points for $n \leqslant 4$.

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* For $n \leqslant 4$, we shall refer to the points P of $\mathfrak{M}_n$ at which the rank is less than maximum as exceptional although, for $n \leqslant 4$, they are not singular in the sense of algebraic geometry.

Lemma 1.

Let $f(z_1, \dots z_n)$ be analytic in the tube and invariant under the orthochronous real Lorentz group, $L^{\uparrow}$. Then, f is also invariant under the complex Lorentz group $\mathfrak{L}$, as long as $\Lambda z_1, \dots \Lambda z_n$ is in the tube. When $\Lambda z_1, \dots \Lambda z_n$ lies out of the tube, the relation $f(\Lambda z_1, \dots \Lambda z_n) = f(z_1 \dots z_n)$ defines a single valued analytic continuation of f to the extended tube.

Proof.

Let $z_1, \dots z_n$ be a fixed point of the tube. Then, for all Λ in a suitable neighbourhood of the identity in $\mathfrak{L}$, $\Lambda z_1, \dots \Lambda z_n$ again lies in the tube. In some sub-neighbourhood, N, we can introduce canonical coordinates $\lambda_1, \dots \lambda_6$ such that[4]

1. As Λ runs over N, $\lambda_1, \dots \lambda_6$ vary over a neighbourhood N' of the origin in the complex six-dimensional Euclidean space with (complex) coordinates $\lambda_1, \dots \lambda_6$.
2. The subset of N, for which $\Lambda \varepsilon L^{\uparrow}$, is the subset of N' for which the $\lambda_1, \dots \lambda_6$ are real.
3. The matrix elements $\Lambda^{\mu}{}_{\nu}$ (and therefore the vector components $\sum\limits_{\nu=0} \Lambda^{\mu}{}_{\nu} z^{\nu}$) are analytic functions of $\lambda_1, \dots \lambda_6$.

Since an analytic function of analytic functions is again analytic, $f(\Lambda z_1, \dots \Lambda z_n)$ is an analytic function of $\lambda_1, \dots \lambda_6$ in N'. Furthermore it has the property that for *real* $\lambda_1, \dots \lambda_6$ it is constant. Therefore it is also constant for complex $\lambda_1, \dots \lambda_6$ in N'[5].

Thus, for $\Lambda \varepsilon N \subset \mathfrak{L}$, equation (1) is satisfied.

This result can be extended immediately in two ways. First, the argument applies when Λ runs over the neighbourhood $I_s N$ of the space inversion, $\Lambda = I_s$. Second, (1) also holds if $\Lambda z_1, \dots \Lambda z_n$ can be connected to $z_1, \dots z_n$ by a curve

$$\Lambda(t) z_1, \dots \Lambda(t) z_n;\ 0 \leqslant t \leqslant 1;\ \Lambda(0) = 1;\ \Lambda(1) = \Lambda,$$

lying entirely within the tube and such that it can be covered by a finite number of overlapping neighbourhoods: $\Lambda(t_j) N z_1, \dots \Lambda(t_j) N z_n$, *lying within the tube.*

However, this last argument by no means completes the proof of (1), because it is not clear that all pairs of points $z_1, \dots z_n$ and $\Lambda z_1, \dots \Lambda z_n$, each of which is in the tube, can be connected by a curve of the sort described above. (If Λ is improper, it is $I_s z_1, \dots I_s z_n$ and $\Lambda z_1, \dots \Lambda z_n$, which have to be connected by the curve. For simplicity of statement, we consider only proper Λ in the rest of this proof. The extension to improper Λ is trivial). We shall give an explicit construction of such a curve at the end of the proof of this Lemma. Assuming the construction for the present, we have completed the proof of the first statement of the Lemma.

The existence of curves of the above described type is closely connected with the possibility of making a single valued analytic continuation of $f(z_1, \dots z_n)$ to the extended tube. For, starting from a fixed point $z_1, \dots z_n$ of the tube, we can extend the analytic function $f(\Lambda z_1, \dots \Lambda z_n)$ of Λ over the whole complex Lorentz group, $\mathfrak{L}$. (It is the simplest possible analytic function on $\mathfrak{L}$, a constant.) f is then defined for points $\Lambda z_1, \dots \Lambda z_n$ of the extended tube. Starting from a different point $z_1' \dots z_n'$ of the tube, f can be defined for the points $M z_1', \dots M z_n'$, $M \varepsilon \mathfrak{L}$. If it happens that for some Λ and M, $\Lambda z_j = M z_j'$, $j = 1, \dots n$, the single valuedness of the extension of f would be insured by: $f(z_1, \dots z_n) = f(z_1', \dots z_n') = f(M^{-1} \Lambda z_1, \dots M^{-1} \Lambda z_n)$. It is just this identity which is guaranteed by our postponed construction of curves, and therefore f as extended is single valued.

The analyticity of f in the extended tube at $\Lambda z_1, \dots \Lambda z_n$ follows from its analyticity at $z_1, \dots z_n$ in the tube, because the partial derivatives at $\Lambda z_1, \dots \Lambda z_n$ are expressible in terms of partial derivatives at $z_1, \dots z_n$, e.g.,

$$\frac{\partial f}{\partial (\Lambda z_1)_\mu}(\Lambda z_1, \dots \Lambda z_n) = \sum_{\nu=0}^{3} \frac{\partial f(z_1, \dots z_n)}{\partial (z_1)^\nu} \frac{\partial (z_1)^\nu}{\partial (\Lambda z_1)_\mu}.$$

This completes the proof of the second statement of the Lemma.

It remains to construct a curve $\Lambda(t) z_1, \dots \Lambda(t) z_n$; $0 \leqslant t \leqslant 1$, $\Lambda(t) \varepsilon \mathfrak{L}$, beginning at an arbitrary point of the tube $z_1, \dots z_n$, ending at the point $\Lambda z_1, \dots \Lambda z_n$ of the tube, and lying entirely within the tube. The existence of such a curve is obvious if Λ is a *real* (orthochronous, proper) Lorentz transformation, because every such transformation leaves the tube invariant and their set is connected.

For Λ complex, the required calculations are simpler in a two dimensional matrix formalism in which the four vector z^μ is represented by the matrix

$$Z = \begin{pmatrix} z^0 + z^3 & z^1 - i z^2 \\ z^1 + i z^2 & z^0 - z^3 \end{pmatrix} \equiv \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}. \tag{4}$$

Then the most general proper complex Lorentz transformation is of the form $z^\mu \to z'^\mu$, where the four vector z'^μ belongs to a matrix, Z', given by

$$Z' = AZB^*. \tag{5}$$

Here A and B are 2×2 matrices of determinant one. In particular, the most general (orthochronous, proper) real Lorentz transformation is of this form with $A = B$. This last fact permits us to simplify our problem. Note that (5) can be written

$$Z' = (AB^{-1})BZB^*, \tag{6}$$

so that the most general complex Lorentz transformation is of the form of a real Lorentz transformation followed by a complex Lorentz transformation of the special form

$$Z' = CZ. \tag{7}$$

It therefore suffices to consider complex Lorentz transformations of this special form. The problem can be simplified further by making a suitable real Lorentz transformation of the final vectors

$$Z' \to DZ'D^* = (DCD^{-1})\, DZD^*.$$

By suitable choice of D, we can bring C into triangular form. In fact, unless the proper values of C are equal, C can be diagonalized. Thus, we can restrict our attention to C of the form

$$\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \tag{8}$$

and

$$\begin{pmatrix} \pm 1 & \tau \\ 0 & \pm 1 \end{pmatrix}. \tag{9}$$

We shall prove that, if Z is in the tube and $Z' = CZ$ is in the tube where C is of the form (8) or (9), then $C(t)Z$ is in the tube where

$$C(t) = \begin{pmatrix} \exp[t(\varrho + i\theta)] & 0 \\ 0 & \exp[-t(\varrho + i\theta)] \end{pmatrix}, \quad \left. \begin{array}{c} 0 \leqslant t \leqslant 1, \\ \zeta = \exp(\varrho + i\theta), \end{array} \right\} \tag{10}$$

$$C(t) = \begin{pmatrix} \pm 1 & t\tau \\ 0 & \pm 1 \end{pmatrix}, \quad 0 \leqslant t \leqslant 1, \tag{11}$$

respectively. Thus, it will be possible to choose the same curve independent of the point Z. (Actually, it will be seen that the case of the minus sign in (11) can be excluded.)

The conditions which express in terms of the matrix Z' the fact that the vector $z'_\mu = \xi'_\mu - i\eta'_\mu$ lies in the tube are

$$-4(\eta')^2 = \det(Z' - Z'^*) < 0, \tag{12}$$

$$\eta'_0 = -\frac{1}{2}\,\mathrm{Im}\, tr(Z') > 0, \tag{13}$$

as can be derived by a simple computation. Our procedure will be to determine all ζ and τ in (8) and (9) consistent with (12) and (13) for fixed z^μ in the tube. We shall see that, if ζ and τ are consistent, so are ζ^t and $t\tau$ for $0 \leqslant t \leqslant 1$.

Consider first the case (9). Then, the condition (12) reads

$$0 > -4(\eta')^2 = \det\begin{pmatrix} \pm(Z_{11} - \bar{Z}_{11}) + \tau Z_{21} - \bar{\tau} Z_{21}, & \pm(Z_{12} - \bar{Z}_{21}) + \tau Z_{22} \\ \pm(Z_{21} - \bar{Z}_{12}) - \bar{\tau}\bar{Z}_{22}, & \pm(Z_{22} - \bar{Z}_{22}) \end{pmatrix}$$

$$= |\tau|^2 |Z_{22}|^2 \mp (s\bar{\tau} + \bar{s}\tau) + \det(Z - Z^*),$$

$$\text{where} \quad s = \bar{Z}_{21} Z_{22} - \bar{Z}_{22} Z_{12}.$$

This inequality can be rewritten in the form

$$|\tau \mp |Z_{22}|^{-2} s|^2 - [|Z_{22}|^{-4}|s|^2 + |Z_{22}|^{-2}(-\det(Z - Z^*))] < 0$$

which describes the interior of a circle in the τ plane about the point

$$\pm |Z_{22}|^{-2} s.$$

As far as the condition (13) is concerned, we note that it is satisfied by all points in the interior of the circle if it is satisfied by any one, because the vector η' must pass through a vector of zero length in order to change the sign of η'_0. With the plus sign in (9), the condition (13) is always satisfied and never for the minus sign, as one sees by considering the case $\tau = 0$.

Evidently, if the point τ is in the interior of the allowed circle, the points $t\tau$, $0 \leqslant t \leqslant 1$ will also be because the origin is in the circle and the circle is convex.

Now we turn to the case (8) in which C is diagonal. Here, there are also two two-dimensional domains of C's consistent with condition (12), one of which is excluded by (13), as we shall see by a detailed consideration. The boundary of the allowed domain is convex in terms of the parameters ϱ and θ defined in (10). This will have to be proved by a detailed computation since the boundary is not an elementary curve.

Condition (12) for the case (8) is

$$\begin{aligned} 0 > -4(\eta')^2 = {} & |Z_{12}|^2 |\zeta|^2 + |Z_{21}|^2 |\zeta|^{-2} \\ & - (Z_{11}\bar{Z}_{22}\zeta\bar{\zeta}^{-1} + \bar{Z}_{11} Z_{22}\bar{\zeta}\zeta^{-1}) + 2\, Re(\det Z). \end{aligned} \tag{14}$$

This condition is satisfied for $-\zeta$ if it is satisfied for ζ and, in particular, since by assumption it is satisfied for $\zeta = +1$, it is also satisfied for $\zeta = -1$. For $\zeta = -1$, the vector z' is not in the tube. Consequently, by the same type of continuity argument which we used in connection with equation (9), there must be at least two disconnected sets of ζ satisfying (14), one of which does not satisfy (13). Another consequence of (13), which we shall use in the following, is $Z_{11} \bar{Z}_{22} \neq 0$.

We divide the remainder of the argument into three parts.

Case 1.

$$Z_{12} \neq 0, \; Z_{21} \neq 0.$$

In this case, we can divide (14) by $2|Z_{12}|\,|Z_{21}|$, and introduce the new variables

$$\left.\begin{aligned} &\sigma_0 = \arg(\bar{Z}_{11} Z_{22}), \; \zeta = \exp(\varrho + i\theta), \\ &\chi = 2\varrho - ln(|Z_{21}|\,|Z_{12}|^{-1}), \; \psi = 2\theta - \sigma_0, \\ &\nu = -|Z_{21}|^{-1}|Z_{12}|^{-1} Re(\det Z), \; \mu = |Z_{11}||Z_{22}||Z_{12}|^{-1}|Z_{21}|^{-1} > 0. \end{aligned}\right\} \tag{15}$$

Equation (14) then reads

$$0 > P = \cosh\chi - (\mu\cos\psi + \nu). \tag{16}$$

Since P is periodic in ψ, it suffices to consider (16) in the strip $|\psi| \leq \pi$, $-\infty < \chi < \infty$, and show that it defines a convex region there.

There are two conditions on the coefficients μ and ν:

$$|\nu| \leqslant 1 + \mu, \tag{17}$$

and

$$\nu + \mu > 1. \tag{18}$$

The first of these merely says the real part of the determinant of a matrix is less in absolute value than the sum of the absolute values of the terms which comprise the determinant. The second is a consequence of the fact that for $\zeta = 1$, $P < 0$ by assumption, and therefore $\nu + \mu\cos\sigma_0 > 1$.

From (17) and (18) it follows that there exists an angle ψ_0, $0 < \psi_0 \leqslant \pi$, such that

$$\nu + \mu\cos\psi_0 = 1.$$

For

$$|\psi| \leqslant |\psi_0|, \; \nu + \mu\cos\psi \geqslant 1. \tag{19}$$

The expression (16) for P makes it clear that its behaviour in χ for fixed ψ is: if $\mu\cos\psi + \nu < 1$, $P > 0$ for all χ; if $\mu\cos\psi + \nu = 1$, $P = 0$ for $\chi = 0$ and $P > 0$

for $\chi \neq 0$; if $\mu \cos\psi + \nu > 1$, $P = 0$ for some $\chi_1 = \chi_1(\psi) > 0$ (and also for $\chi = -\chi_1$), $P < 0$ for $|\chi| < \chi_1$, and $P > 0$ for $|\chi| > \chi_1$.

The discussion of the preceding two paragraphs shows that the set of χ and ψ satisfying $P < 0$ in the strip $|\psi| \leqslant \pi$, $-\infty < \chi < \infty$, is connected and invariant under reflections in the $\chi = 0$ and $\psi = 0$ axes, being the domain:' $-\chi_1(\psi) < \chi < \chi_1(\psi)$, $-\psi_0 < \psi < \psi_0$, where χ_1 is given by

$$\cosh \chi_1 = \mu \cos\psi + \nu, \quad \chi_1 > 0. \tag{20}$$

To complete the argument we will show the convexity of the function χ_1 as a function of ψ.

Differentiating (20) twice and eliminating the first derivative, χ_1', of χ with respect to ψ, we find, for $|\psi| < \psi_0$, $(\sinh \chi_1)^3 \chi_1'' = -\mu Q$, where

$$Q = \cos\psi (\sinh \chi_1)^2 + \mu (\sin\psi)^2 \cosh \chi_1 \tag{21}$$

$$= (\mu \cos\psi + \nu)(\mu + \nu \cos\psi) - \cos\psi, \tag{22}$$

or

$$Q = \mu\nu (\cos\psi + 1)^2 - [1 - (\mu - \nu)^2] \cos\psi. \tag{23}$$

We assert that $Q > 0$ for $|\psi| < \psi_0$. For $0 \leqslant \psi \leqslant \frac{\pi}{2}$, this is an immediate consequence of (21), since both terms on the right hand side of (21) are positive there. For $1/2\,\pi < \psi < \psi_0$, we have $\cos\psi_0 < 0$, and from (19), $\nu = 1 + \mu |\cos\psi_0|$. Using this last fact, we see that, when $\mu > \nu$, $\mu \cos\psi + \nu > 0$, $\mu + \nu \cos\psi > 0$ and $-\cos\psi > 0$ so that (22) immediately implies $Q > 0$. Again, when $\mu \leqslant \nu$, we use the form (23), and note that $1 - (\mu - \nu)^2 = (1 - \mu + \nu)(1 + \mu - \nu)$, and $1 + \nu - \mu > 0$, and $1 + \mu - \nu = \mu(1 - |\cos\psi_0|) \geqslant 0$, so $Q > 0$ in this case too.

Case II.

$$Z_{12} = 0, \; Z_{21} \neq 0 \; (\text{or } Z_{12} \neq 0, \; Z_{21} = 0).$$

Here we define σ_0, ϱ, θ, and ψ as in (15), but μ, ν, χ as follows

$$\chi = -2\varrho + ln\left(\frac{1}{2}|Z_{21}|^2\right), \quad \mu = |Z_{11}||Z_{22}| > 0, \quad \nu = -Re\,(\det Z). \tag{24}$$

Then, the basic inequality (14) takes the form

$$0 > P = e^{\chi} - (\mu \cos\psi + \nu), \tag{25}$$

and the analogues of the inequalities (17) and (18) are

$$|\nu| \leqslant \mu \tag{26}$$

and

$$\nu + \mu > 0. \tag{27}$$

(If $Z_{12} \neq 0$, $Z_{21} = 0$, then, in equation (24), replace the definition of χ by $\chi = 2\,\varrho + ln\left(\frac{|Z_{12}|^2}{2}\right)$. This case is essentially the same as that for $Z_{12} = 0$ and $Z_{21} \neq 0$ and will not be discussed further.)

A discussion analogous to that in Case I shows that the region of the strip $|\psi| \leqslant \pi$, $-\infty < \chi < \infty$, as determined by $P < 0$, is given by

$$-\infty < \chi < \chi_1(\psi), \quad |\psi| < \psi_0 \leqslant \pi,$$

$$\mu \cos \psi_0 + \nu = 0, \text{ and } e^{\chi_1} = \mu \cos \psi + \nu.$$

Here, $\exp(2\,\chi_1)\,\chi_1'' = -\mu Q$, where $Q = \mu + \nu \cos \psi > 0$, so that the region is convex.

Case III.

$$Z_{12} = Z_{21} = 0.$$

Here the basic inequality is

$$0 > P = -(\mu \cos \psi + \nu),$$

with μ, ψ and ν defined as in Case II. μ and ν satisfy the same inequalities as in Case II, namely (26) and (27), so that there again exists a ψ_0 satisfying $\mu \cos \psi_0 + \nu = 0$, $0 < \psi_0 \leqslant \pi$, and the region permitted by $P < 0$ is the strip $|\psi| < \psi_0$, $-\infty < \chi < \infty$ which is obviously convex.

In each of these three cases we have proved that the region of χ, ψ space (or what is essentially the same thing since it is obtained by a translation and change of scale, ϱ, θ space) permitted by conditions (12) and (13) is convex. Since these regions contain the point $\varrho = 0$, $\theta = 0$, they also contain the points $t\varrho$, $t\theta$, $0 \leqslant t \leqslant 1$, corresponding to the transformations (10) and (11); so the proof of Lemma 1 is complete.

Lemma 2.

Let $z_1, \dots z_n$ and $\zeta_1, \dots \zeta_n$ be any two sets of n vectors such that

$$Z_{ij} = z_i \cdot z_j = \zeta_i \cdot \zeta_j, \quad i, j = 1, \dots n. \tag{28}$$

If the rank of the $n \times n$ matrix Z is three or four (or, for $n \leqslant 2$ if Z is non-singular), then there exists a complex Lorentz transformation, Λ, such that

$$\Lambda z_i = \zeta_i, \; i = 1, \dots n. \tag{29}$$

If the rank of Z is two or one (and $n > 2$ or $n > 1$, respectively), a $\Lambda \varepsilon \mathfrak{L}$ satisfying (29) will not exist, in general, but there always exists a Λ satisfying

$$\Lambda z_i = \zeta_i + \alpha_i \omega, \tag{30}$$

where α_i are complex numbers and ω is a vector of zero length orthogonal to ζ_i and Λz_i, $i = 1, \dots n$.

Proof.

We note the known fact that, for a symmetric matrix, the rank determined from principal minors is the same as the rank determined from all minors[6]. Thus, if the rank of Z is r, there exist r vectors, say $z_1, \dots z_r$, which have non-vanishing Gram determinant

$$0 \neq G(z_1, \dots z_r) \equiv \det (z_i \cdot z_j), \; i, j = 1, \dots r.$$

This result will be used tacitly many times in the following.

The first step in the proof is to establish the connection between the condition, $G(z_1 \dots z_r) \neq 0$, and the linear independence of the set of vectors $z_1 \dots z_r$. If $G(z_1 \dots z_r) \neq 0$, then the set $z_1 \dots z_r$ is linearly independent. For a relation

$$\sum_{j=1}^{r} \alpha_j z_j = 0 \tag{31}$$

would imply

$$\sum_{j=1}^{r} \alpha_j (z_k \cdot z_j) = 0 \qquad k = 1 \dots r \tag{32}$$

and these last equations have a non-trivial solution $\alpha_1 \dots \alpha_r$ if and only if $\det(z_j \cdot z_k) = 0$, $j, k = 1 \dots r$. The converse, that the linear independence of the set $z_1 \dots z_r$ implies $G(z_1 \dots z_r) \neq 0$, is not true in general. For example,

$$z_1 = (0, 1, 0, 0), \qquad z_2 = (1, 1, 1, 0)$$

is a pair of linearly independent vectors having zero Gram determinant. However, for $r = 4$, the converse holds, for $G(z_1 \dots z_4) = 0$ implies that the equations (32) have a non-trivial solution α_1, a_2, α_3, α_4. This, in turn, implies that there is a vector of the form $\sum_{j=1}^{4} \alpha_i z_j$, with at least one $\alpha_j \neq 0$, which is orthogonal to all z_j. If the z_j were linearly independent, this last would be impossible, since $\sum_{j=1}^{4} \alpha_j z_j$ would then be orthogonal to every vector. Thus, $G(z_1 \dots z_4) = 0$ implies that the set $z_1 \dots z_4$ is linearly dependent.

Next, we show that, under certain circumstances, we can confine our attention to the case $n \leqslant 4$. Let $z_1, \dots z_n$ and $\zeta_1, \dots \zeta_n$ be two sets of vectors such that any $r+1$ element subset of either is linearly dependent, and the matrices $z_j \cdot z_k$ and $\zeta_j \cdot \zeta_k$, $j, k = 1, 2, \dots n$, are equal and of rank r. For convenience, we may suppose that $G(z_1, \dots z_r) \neq 0$. Expand the z_j, $j = r+1, \dots n$ in terms of the z_j, $j = 1, \dots r$.

$$z_j = \sum_{l=1}^{r} \alpha_{jl} z_l, \quad j = r+1, \dots n. \tag{33}$$

The α_{jl} are expressible in terms of scalar products since they are the solutions of the linear equations

$$z_k \cdot z_j = \sum_{l=1}^{r} \alpha_{jl} z_k \cdot z_l, \quad \begin{array}{l} j = r+1, \dots n \\ k = 1, \dots r. \end{array} \tag{34}$$

The equations (34) have a unique solution because $G(z_1, \dots z_r) \neq 0$. Thus, we see that, if a Lorentz transformation Λ can be found satisfying (29) or (30) for $j = 1, \dots r$, it will also satisfy them for $j = r+1, \dots n$, provided that the rank of $z_i \cdot z_j = \zeta_i \cdot \zeta_j$ is r and there are at most r linearly independent z_i and ζ_i. These last provisos are *always* satisfied if $r = 4$ or 3. For $r = 4$, we have just established the equivalence of $G(z_1 \dots z_4) \neq 0$ and linear independence of the vectors z_1, z_2, z_3, z_4. By the very same argument, it cannot happen that the rank of $z_i \cdot z_j$ is three and the number of linearly independent z_j is four. The fact that there can be "extra" linearly independent vectors when the rank of $z_i \cdot z_j$ is one or two is the source of the possibility that (30), but not (29), may hold.

Now we will construct a Λ satisfying (29) under the assumption of the preceding paragraph, i. e., that there is at most a linearly independent set of r z's and at most a linearly independent set of r ζ's and the matrix $z_i \cdot z_j = \zeta_i \cdot \zeta_j$ has rank r. Our preceding considerations assure us that it suffices to consider the case $n \leqslant 4$. The z_i, $i = 1, \dots n$ span an r-dimensional linear manifold, M. For convenience, we let the $z_1, \dots z_r$ be a linearly independent set. Let the corresponding r-dimensional manifold spanned by the ζ_i, $i = 1, \dots n$ be devoted by N. The orthogonal manifolds $M^\perp$ and $N^\perp$ respectively, are $4-r$ dimensional and the intersections $M \cap M^\perp$ and $N \cap N^\perp$ contain only the vector zero. A proof of these last statements is obtained as follows. Supplement the vectors $z_1, \dots z_r$ by $z'_1, \dots z'_{4-r}$ and the vectors $\zeta_1, \dots \zeta_r$ by $\zeta'_1, \dots \zeta'_{4-r}$ so that the resulting sets are bases for the whole four-dimensional space. Then the r equations

$$\sum_{l=1}^{r} \alpha_l (z_l \cdot z_j) + \sum_{l=1}^{4-r} \alpha_{r+l} (z_l' \cdot z_j) = 0 \quad j = 1, \dots r,$$

have rank r, so they have $4-r$ linearly independent solutions

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

Because the basis vectors are linearly independent, the vectors

$$\sum_{l=1}^{r} \alpha_l z_l + \sum_{l=1}^{4-r} \alpha_{l+r} z_l'$$

constitute a linearly independent $4-r$ element set. Consequently, $M^\perp$ has dimension $4-r$. That $M \cap M^\perp$ contains no non-zero vector is equivalent to the statement that the equations

$$\sum_{l=1}^{r} \alpha_l (z_l \cdot z_j) = 0 \quad j = 1, \dots r \tag{35}$$

have no non-trivial solution. The analogous statements for $N^\perp$ and $N \cap N^\perp$ are obtained by replacing z_i by ζ_i in the above proof.

Notice that the Gram determinant of the entire basis

$$z_1, \dots z_r, z_1', \dots z_{4-r}'$$

is the product of the Gram determinants of the sets $z_1, \dots z_r$ and $z_1', \dots z_{4-r}'$ so that the Gram determinant of $z_1', \dots z_{4-r}'$ is non-zero. A similar statement holds for the $\zeta_1', \dots \zeta_{4-r}'$, and we want to use these facts to show that new bases $z_1'', \dots z_{4-r}''$ and $\zeta''_1, \dots \zeta_{4-r}''$ for $M^\perp$ and $N^\perp$, respectively, can be chosen so that

$$z_i'' \cdot z_j'' = \delta_{ij} = \zeta_i'' \cdot \zeta_j'' \quad i, j = 1, \dots 4-r. \tag{36}$$

Consider $M^\perp$. Since the Gram determinant of the $z_1', \dots z_{4-r}'$ does not vanish, some scalar product of these vectors does not vanish, and, consequently, there is at least one vector of non-zero length in $M^\perp$. Adjust its length to 1 and call it z_1''. By induction, using the arguments of this and the immediately preceding paragraph, we can construct $z_1'', z_2'', \dots z_{4-r}''$ orthogonal to each other and of length one. An analogous construction holds for the $\zeta_1'', \zeta_2'', \dots \zeta_{4-r}''$. It is evidently crucial for the success of the construction that at each stage the relevant Gram determinants are non-zero[7].

Now we are in a position to define the Λ required by (29) as the complex linear transformation determined by the equations

$$\begin{aligned} \Lambda z_j &= \zeta_j \qquad & j &= 1, \ldots r, \\ \Lambda z_j'' &= \zeta_j'' \qquad & j &= 1, \ldots 4-r. \end{aligned}$$

Λ so defined preserves scalar products by virtue of (35) and (36):

$$\left.\begin{aligned} &\left[\Lambda\left(\sum_{j=1}^{r}\alpha_j z_j + \sum_{j=1}^{4-r}\alpha_{j+r} z_j''\right)\right]\cdot\left[\Lambda\left(\sum_{j=1}^{r}\beta_j z_j + \sum_{j=1}^{4-r}\beta_{j+r} z_j''\right)\right] \\ &= \sum_{j,k=1}^{r}\alpha_j\beta_k(\Lambda z_j)\cdot(\Lambda z_k) + \sum_{j,k=1}^{4-r}\alpha_{j+r}\beta_{k+r}(\Lambda z_j'')\cdot(\Lambda z_k'') \\ &= \sum_{j,k=1}^{r}\alpha_j\beta_k\zeta_j\cdot\zeta_k + \sum_{j,k=1}^{4-r}\alpha_{j+r}\beta_{k+r}\zeta_j''\cdot\zeta_k'' \\ &= \sum_{j,k=1}^{r}\alpha_j\beta_k z_j\cdot z_k + \sum_{j,k=1}^{4-r}\alpha_{j+r}\beta_{j+r} z_j''\cdot z_k'' \\ &= \left(\sum_{j=1}^{r}\alpha_j z_j + \sum_{j=1}^{4-r}\alpha_{j+r} z_j''\right)\cdot\left(\sum_{j=1}^{r}\beta_j z_j + \sum_{r=1}^{4-r}\beta_{j+r} z_j''\right). \end{aligned}\right\} \quad (37)$$

It is therefore the required Lorentz transformation, and the proof of Lemma 2 is complete for the rank three and four cases.

For rank two and one we must deal with the cases in which the number of linearly independent z_j (or ζ_j) is larger than the rank of the matrix $z_j \cdot z_k$. That we cannot expect to find a Λ satisfying (29) in this case is clear from the example $z_1 = (1, 0, 0, 0)$, $z_2 = (1, 1, i, 0)$; $\zeta_1 = (1, 0, 0, 0)$, $\zeta_2 = (1, 0, 0, 0)$ of vectors satisfying (28) with a matrix of rank one; a Λ certainly cannot carry linearly independent vectors (the z's) into linearly dependent vectors (the ζ's).

Let Z be of rank $r = 1$, or 2, and let $n > r$. There is a subset of r vectors z_i with non-vanishing Gram determinant. They span a subspace which we call M_1. It is a subspace of M, the subspace spanned by all z_i, $i = 1, \ldots n$, whose dimension we denote by m. The corresponding subspaces for the vectors ζ_i, we denote by N_1 (of dimension r) and N (of dimension m'). Because M_1 and N_1 have non-vanishing Gram determinants there is a unique decomposition of the vectors z_i and ζ_i:

$$\begin{aligned} z_i &= z_i' + z_i'' \quad & i &= 1, \ldots n,\ z_i' \in M_1,\ z_i'' \in M_1^{\perp} \cap M \\ \zeta_i &= \zeta_i' + \zeta_i'' \quad & i &= 1, \ldots n,\ \zeta_i' \in N_1,\ \zeta_i'' \in N_1^{\perp} \cap N. \end{aligned}$$

Now the rank of the matrices $z_i'' \cdot z_j''$, $\zeta_i'' \cdot \zeta_j''$, $i, j = 1, \ldots n$ must be zero, since the rank of $z_i' \cdot z_j'$ and $\zeta_i' \cdot \zeta_j'$ is already r. Furthermore, the subspaces $M_1^{\perp} \cap M$ and $N_1^{\perp} \cap N$ of the $4-r$ dimensional spaces $M_1^{\perp}$, and $N_1^{\perp}$ can at most be one-dimensional since they are isotropic and $M_1^{\perp}$ and $N_1^{\perp}$ have non-vanishing Grammian[7]. Thus, $z_i'' = \gamma_i \omega'$, $i = 1, \dots n$, $\gamma_1, \dots \gamma_n$ complex numbers and $(\omega')^2 = 0$, $\omega' \cdot z_i' = 0$, $i = 1, \dots n$, and similarly $\zeta_i'' = \delta_i \omega$, $\omega^2 = 0 = \omega \cdot \zeta_i'$, $i = 1, \dots n$. Incidentally, we see that the dimensions m and m' of M and N are $r+1$ at most.

Now we choose an orthonormal basis ω_i, $i = 1, \dots 4-r$ for $M_1^{\perp}$ and η_i, $i = 1, \dots 4-r$ for $N_1^{\perp}$ so that ω' and ω lie in the subspaces spanned by ω_1, ω_2 and η_1, η_2 respectively. For $r = 2$, no construction is required to obtain this property of ω's and η's. For $r = 1$, we know the construction is possible because there has to be at least one vector of non-zero length orthogonal to ω' (or ω) and we can take it suitably normalized to be ω_3 (or η_3). Since ω' is a linear combination of ω_1 and ω_2 and of zero length, it must be of the form $a(\omega_1 \pm i\omega_2)$ and by changing the sign of ω_2, if necessary, we can arrange it so that $\omega' = a(\omega_1 + i\omega_2)$. Similarly, $\omega = b(\eta_1 + i\eta_2)$.

Finally, the Lorentz transformation Λ required by (30) is the linear transformation defined by

$$\begin{aligned} \Lambda z_i' &= \zeta_i' \quad \text{for the } r \text{ vectors } z_i' \text{ and } \zeta_i' \text{ which span } M_1 \text{ and } N_1 \\ \Lambda \omega_i &= \eta_i \quad i = 1, 2, \dots, 4-r. \end{aligned}$$

That this is indeed a Lorentz transformation follows by an argument like that used for the higher ranks. That Λ satisfies (30) follows from the computation

$$\begin{aligned} \Lambda z_k &= \Lambda z_k' + \gamma_k a(\Lambda\omega_1 + i\Lambda\omega_2) \\ &= (\zeta_k - \delta_k \omega) + \gamma_k a(\eta_1 + i\eta_2) \\ &= \zeta_k + (\gamma_k a - \delta_k b)(\eta_1 + i\eta_2) \qquad k = 1, \dots n. \end{aligned}$$

Therefore, when the rank r of $z_i \cdot z_j$ is one or two, there exists a Lorentz transformation, Λ, such that the Λz_i and ζ_i differ by multiples of a fixed vector orthogonal to all Λz_i and ζ_i, i = 1, . . . n, and of length zero. This completes the proof.

To round out the information provided by Lemma 2, we make three additional remarks. First of all, if Z is an arbitrary complex symmetric $n \times n$ matrix of rank $r \leqslant 4$, it can be written in the form

$$Z_{ij} = \zeta_i \cdot \zeta_j, \; i, j = 1, \dots n, \tag{38}$$

where $\zeta_1, \dots \zeta_n$ are four vectors which span a linear manifold M of dimension r. This follows immediately from the standard theorem of algebra which says that, if Z is a complex symmetric $n \times n$ matrix of rank r, there exists a non-singular $n \times n$ matrix S such that

$$Z = S \left\{ \begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array} \right\} S^{\mathrm{T}}, \tag{39}$$

where the one in the upper left hand corner of the matrix indicated in curly brackets stands for the $r \times r$ unit matrix[8]. One can then take the components ζ_j^{μ} as $S_{j1}, iS_{j2}, \dots iS_{jr}$ followed by $4-r$ zeros. Equation (39) then reduces to (38). At most r of these vectors can be linearly independent and, in fact, exactly r are because otherwise Z would have rank less than r.

Our second remark is that, if the point $\zeta_1+\alpha_1\omega, \dots \zeta_n+\alpha_n\omega$ is in the tube and $\omega^2 = 0 = \omega\cdot\zeta_j$, $j = 1, \dots n$, then ω and ζ_j are of the form

$$\omega = a(\omega_1 + i\omega_2), \text{ where } \omega_1 \text{ and } \omega_2 \text{ are real and}$$

$$\omega_1^2 = \omega_2^2 = -1, \; \omega_1\cdot\omega_2 = 0,$$

$$\zeta_j = \zeta_j' + \beta_j\,\omega, \text{ where } \zeta_j'\cdot\omega_1 = 0 = \zeta_j'\cdot\omega_2, j = 1, \dots n,$$

$$\text{and } \zeta_1' \dots \zeta_n' \text{ is a point of the tube.}$$

To prove these statements, we split ω into its real and imaginary parts: $\omega = q + ir$. Then, $\omega^2 = 0$ implies $q^2 = r^2$ and $q\cdot r = 0$ so that q and r are either light-like and collinear or space-like and orthogonal. The first alternative cannot occur, because the requirement $\omega\cdot\zeta_j = 0$ would then force q and r to be orthogonal to a time-like vector, and the first half of (40) follows if we choose ω_1 and ω_2 as q and r normalized to length minus one. The second half is easily seen if the real and imaginary parts of ζ_j are expanded in terms of ω_1 and ω_2 as follows:

$$\left.\begin{aligned} \xi_j &= \varrho_1^{(j)}\omega_1 + \varrho_2^{(j)}\omega_2 + \xi_j', \text{ where } \xi_j'\cdot\omega_1 = 0 = \xi_j'\cdot\omega_2, \; j = 1, \dots n, \\ \eta_j &= \sigma_1^{(j)}\omega_1 + \sigma_2^{(j)}\omega_2 + \eta_j', \text{ where } \eta_j'\cdot\omega_1 = 0 = \eta_j'\cdot\omega_2, \; j = 1, \dots n. \end{aligned}\right\} \tag{40}$$

The orthogonality condition $(\xi_j - i\eta_j)\cdot\omega = 0$ then leads directly to $\sigma_1^{(j)} = \varrho_2^{(j)}$ and $\sigma_2^{(j)} = -\varrho_1^{(j)}$, and therefore

$$\zeta_j = \xi_j - i\eta_j = \xi_j' - i\eta_j' + (\varrho_1^{(j)} - i\sigma_1^{(j)})\,(\omega_1 + i\omega_2).$$

If we write $\zeta_j' = \xi_j' - i\eta_j'$, it remains to verify that the point $\zeta_1', \dots \zeta_n'$ is in the tube. Because $\zeta_1+\alpha_1, \omega, \dots \zeta_n+\alpha_n\omega$ lies in the tube by assumption, the squares of the lengths of the imaginary parts of $\zeta_j+\alpha_j\omega$, $j = 1, \dots n$ are positive:

$$[\eta_j' - (\beta_j''\omega_1 + \beta_j'\omega_2)]^2 = \eta'^2 - (\beta_j'')^2 - (\beta_j')^2 > 0, \tag{41}$$

where $(\beta_j + \alpha_j)\; a = \beta_j' + i\beta_j''$ with β_j' and β_j'' real, and β_j and a defined in (40). Clearly, (41) implies $\eta_j'^2 > 0$. Furthermore, β_j'' and β_j' can be continuously decreased to zero without (41) losing its validity. Consequently, if the imaginary part of $\zeta_j + \alpha_j\,\omega$ points into the forward cone, so must that of ζ'_j, so that the point $\zeta'_1, \dots \zeta'_n$ is in the tube.

Our third remark is that, if ζ'_j, $j = 1, \dots n$ and ω have the properties $\omega = a\,(\omega_1 + i\,\omega_2)$, with ω_1 and ω_2 real and $\omega^2{}_1 = -1 = \omega_2{}^2$, $\omega_1 \cdot \omega_2 = 0$, $\zeta'_j \cdot \omega_1 = 0 = \zeta'_j \cdot \omega_2$ for $j = 1, \dots n$, then there exists a one-parameter family of Lorentz transformations $\Lambda(\theta)$, $0 \leqslant \theta \leqslant \infty$ with the properties $\Lambda(\theta)\zeta'_j = \zeta'_j$, $j = 1, \dots n$ and $\Lambda(\theta)\,\omega = e^{-\theta}\omega$. The transformation $\Lambda(\theta)$ is defined as the identity on vectors orthogonal to ω_1 and ω_2, but

$$\left.\begin{aligned} \Lambda(\theta)\,\omega_1 &= \omega_1 \cos i\theta - \omega_2 \sin i\theta \\ \Lambda(\theta)\,\omega_2 &= \omega_1 \sin i\theta + \omega_2 \cos i\theta . \end{aligned}\right\} \tag{42}$$

With its definition completed by linearity, $\Lambda(\theta)$ is a Lorentz transformation with the required properties.

Two important consequences follow immediately from these remarks. The first is that, if a point $\zeta_1 + \alpha_1\omega, \dots \zeta_n + \alpha_n\omega$ with $\omega^2 = 0$, $\omega \cdot \zeta_j = 0$, $j = 1, \dots n$ lies in the extended tube, then all points of the form $\zeta_1 + \alpha'_1\omega, \dots \zeta_n + \alpha'_n\omega$ do also, where $\alpha'_1, \dots \alpha'_n$ are arbitrary complex numbers. Since $\zeta_1 + \alpha_1\omega, \dots \zeta_n + \alpha_n\omega$ is in the extended tube, there exists a complex Lorentz transformation Λ such that $\Lambda\zeta_1 + \alpha_1\Lambda\omega, \dots \Lambda\zeta_n + \alpha_n\Lambda\omega$ is in the tube. Using the second remark, we may then write $\Lambda\zeta_j + \alpha'_j\Lambda\omega$ as $\Lambda\zeta'_j + (\beta_j + \alpha'_j)\Lambda\omega$, where $\Lambda\zeta'_1, \dots \Lambda\zeta'_n$ is a point of the tube, and $\Lambda\omega = a\,(\omega_1 + i\,\omega_2)$, $\omega_1^2 = -1 = \omega_2^2$ and $\omega_1 \cdot \omega_2 = \Lambda\zeta'_j \cdot \omega_1 = \Lambda\zeta'_j \cdot \omega_2 = 0$, $j = 1, \dots n$. Using the third remark, we obtain a family of transformations $\Lambda(\theta)$ such that

$$\Lambda(\theta)\,[\Lambda\zeta_j + \alpha'_j\Lambda\,\omega] = \Lambda\zeta'_j + e^{-\theta}(\beta_j + a'_j)\Lambda\omega,\ j = 1, \dots n.$$

These equations say that the point $\Lambda\zeta_1 + \alpha'_1\Lambda\omega, \dots \Lambda\zeta_n + \alpha'_n\Lambda\omega$ can be brought arbitrarily near to the point $\Lambda\zeta'_1, \dots \Lambda\zeta'_n$ (which lies in the tube) by a complex Lorentz transformation. Therefore, the point $\Lambda\zeta_1 + \alpha'_1\Lambda\omega, \dots \Lambda\zeta_1 + \alpha'_n\Lambda\omega$ and, consequently, the point $\zeta_1 + \alpha'_1\omega, \dots \zeta_n + \alpha'_n\omega$ lie in the extended tube.

The second consequence is that an invariant analytic function (satisfying the hypotheses of Lemma 1) is necessarily single-valued on $\mathfrak{M}_n$. For points of $\mathfrak{M}_n$ where the rank of $z_i \cdot z_j$ $i, j = 1, \dots n$ is three or min $(3, n)$ the statement follows immediately, because $f(z_1, \dots z_n) = f(\zeta_1, \dots \zeta_n)$ is a consequence of $z_i \cdot z_j = \zeta_i \cdot \zeta_j$, $i, j = 1, \dots n$, since, by Lemma 2 there exists a $\Lambda \varepsilon \mathfrak{L}$ such that $\Lambda z_j = \zeta_j$, $j = 1, \dots n$. For $z_i \cdot z_j$, $i, j = 1, \dots n$, of rank 2, $n > 2$, or 1, $n > 1$, we know by Lemma 2 that $z_i \cdot z_j = \zeta_i \cdot \zeta_j$ implies $\Lambda z_j = \zeta_j + \alpha_j\,\omega$, $j = 1, \dots n$, so to show the single valuedness of f at such points of $\mathfrak{M}_n$ it suffices to show that

$$f(\zeta_1, \ldots \zeta_n) = f(\zeta_1 + \alpha_1\omega, \ldots \zeta_n + \alpha_n\omega). \tag{43}$$

Furthermore, without loss of generality, we may restrict ourselves to the case in which ζ_j and ω have the properties $\omega = a(\omega_1 + i\omega_2)$ with ω_1 and ω_2 real and $\omega_1^2 = \omega_2^2 = -1$, $\omega_1 \cdot \omega_2 = 0$, and $\zeta_j \cdot \omega_1 = 0 = \zeta_j \cdot \omega_2$, $j = 1, \ldots n$. By introducing the $\Lambda(\theta)$ defined in our third remark above, we get

$$\begin{aligned} &f(\zeta_1, \ldots \zeta_n) - f(\zeta_1 + \alpha_1\omega, \ldots \zeta_n + \alpha_n\omega) \\ &= f(\Lambda(\theta)\zeta_1, \ldots \Lambda(\theta)\zeta_n) - f(\Lambda(\theta)\zeta_1 + \alpha_1\Lambda(\theta)\omega, \ldots \Lambda(\theta)\zeta_n + \alpha_n\Lambda(\theta)\omega) \\ &= f(\zeta_1, \ldots \zeta_n) - f(\zeta_1 + \alpha_1 e^{-\theta}\omega, \ldots \zeta_n + \alpha_n e^{-\theta}\omega). \end{aligned}$$

From the continuity of f at $\zeta_1, \ldots \zeta_n$, we see that the last expression vanishes in the limit as $\theta \to \infty$, which proves (43).

Now we turn to the connection between the topology of the vectors $z_1, \ldots z_n$ and the topology on $\mathfrak{M}_n$.

Lemma 3.

Let Z be an $n \times n$ complex symmetric matrix of rank r, $1 \leqslant r \leqslant 4$ and ω an arbitrary real positive number. Then there exists a set of n four vectors $z_1, \ldots z_n$ and a neighbourhood of them consisting of the four-vectors $z_1 + v_1, \ldots z_n + v_n$ with

$$|v_j^\mu| < \omega \quad j = 1, \ldots n; \ \mu = 0, 1, 2, 3 \tag{44}$$

such that $Z_{ij} = z_i \cdot z_j$, $i, j = 1, \ldots n$ and the matrices, Z'_{ij}, defined by

$$Z'_{ij} = (z_i + v_i) \cdot (z_j + v_j), \ i.j, = 1 \ldots n, \tag{45}$$

cover a neighbourhood of Z in the set of complex symmetric matrices of rank $\leqslant 4$, i.e., for suitably chosen $\eta > 0$, every complex symmetric matrix Z' of rank $\leqslant 4$ which satisfies

$$|Z_{ij} - Z'_{ij}| < \eta \qquad i, j = 1, \ldots n$$

is of the form (45) with v_j satisfying (44).

Proof.

The direct determination of the vectors v_i, $i = 1, \ldots n$ satisfying (45) would be somewhat involved, so we make a series of transformations to reduce the problem to a simpler one.

We know from the first remark following Lemma 2 that the matrix Z may be written as a matrix of scalar products: $Z_{ij} = z_i \cdot z_j$, $i, j = 1, \ldots n$, where, if Z has rank r, the vectors $z_j, j = 1, \ldots n$ span a linear manifold of dimension r. That being the case, there exists an r-element linearly independent subset of the n vectors which may as well, for convenience, be taken as $z_1, \ldots z_r$. A new set of z's, which we denote by $z'_j, j = 1, \ldots n$, is defined by

$$z'_j = z_j \quad j = 1, \ldots r$$

$$0 = z'_j = z_j - \sum_{k=1}^{r} \alpha_{jk} z_k, \quad j = r+1, \ldots n.$$

This linear transformation from z's to z''s has determinant 1 and is therefore non-singular. Subsequent to this transformation, we carry out a linear transformation on the subspace M which normalizes and orthogonalizes the $z'_j, j = 1, \ldots r$. The product of these two transformations is given by a matrix A which has the property

$$AZA^{\mathsf{T}} = \begin{array}{c} \\ r \end{array} \overset{r}{\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array} \right)}.$$

Now, since A is non-singular, it maps neighbourhoods of $z_1, \ldots z_n$ into neighbourhoods of $\sum_{j=1}^{n} A_{1j} z_j, \ldots \sum_{j=1}^{n} A_{nj} z_j$ and neighbourhoods of Z into neighbourhoods of AZA^{T} in an invertible manner. Thus, it suffices to prove Lemma 3, in the case that the first r of the vectors z_j are orthonormal and the rest are zero.

This first simplification of the problem uses a transformation, A, which depends only on the z_j, but not on which point in the neighbourhood of the z_j is under consideration. The second transformation we make will be different for each Z', and makes the first r of the vectors $z_j + v_j$ orthogonal to the rest.

Define a new set of v_j by the equations

$$v'_j = v_j \quad j = 1, \ldots r,$$

$$v'_j = v_j - \sum_{k=1}^{r} \beta_{jk}(z_k + v_k) \quad j = r+1, \ldots n,$$

where the numbers $\beta_{jk}, j = r+1, \ldots n, k = 1, \ldots r$, are determined from the condition

$$v'_j \cdot (z_k + v_k) = 0, \; j = r+1, \ldots n, \; k = 1, \ldots r. \tag{46}$$

An elementary calculation yields

$$\beta_{jl} = \sum_{k=1}^{r} Z'_{jk} (Z'^{-1})_{kl} \qquad j = r+1, \ldots n, \; l = 1, \ldots r,$$

where the indicated matrix inverse means the inverse of the $r \times r$ matrix Z'_{jk}, $j, k = 1, \ldots r$. The simplification of the problem achieved in the preceding paragraph enables us to write $Z' = Z + B$ where $Z_{jk} = \delta_{jk}, j, k = 1, \ldots r$, $Z_{jk} = 0$, $j, k > r$. When the matrix elements of B satisfy $|B_{jk}| < \eta, j, k = 1, \ldots n$ and η is sufficiently small, the estimates $|Z'_{jk}| < \eta; j = r+1, \ldots n, k = 1, \ldots r$, and $|(Z'^{-1})_{kl} - \delta_{kl}| < \eta\,(1 - r\eta)^{-1}$, $k, l = 1, \ldots r$, hold, so that the transformation from the v's to the v''s has an inverse and carries small neighbourhoods of the origin in v space into small neighborhoods in v' space, and vice versa. Thus the problem has been reduced to that of finding $v'_1, \ldots v'_n$ satisfying

$$\left(\begin{array}{c|c} (z_j + v'_j)\cdot(z_k + v'_k) & (z_j + v'_j)\cdot v'_k \\ \hline v'_j\cdot(z_k + v'_k) & v'_j \cdot v'_k \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) + \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right), \qquad (47)$$

where the dividing line in the matrix is at the r^{th} row and column and

$$(B_1)_{jk} = B_{jk}, \quad j, k = 1, \ldots r, \text{ while}$$

$$(B_2)_{jk} = B_{jk} - \sum_{l,\, m=1}^{r} (Z')_{jl} (Z'^{-1})_{lm} (Z')_{mk} \quad j, k = r+1, \ldots n.$$

When $|B_{jk}| < \eta, j, k = r+1, \ldots n$ we have $|(B_2)_{jk}| \leqslant \eta\,(1 - r\eta)^{-1}$.

The simple expedient of requiring v'_k, $k = 1, \ldots r$ to have vanishing components beyond the r^{th} and v'_k, $k = r+1, \ldots n$, vanishing components before the $(r+1)$st in a basis in which the vectors z_k, $k = 1, 2, \ldots r$, are the first r coordinate axes guarantees the orthogonality relations (46). The problem of satisfying the upper left hand corner of the relation (47) then reads in $r \times r$ matrix form

$$(1 + v')(1 + v')^{\mathsf{T}} = 1 + B_1,$$

where $1 + v'$ is the $r \times r$ matrix whose k^{th} row is composed of the components of $z_k + v'_k$. By making the special choice of the components of v'_k, $k = 1, 2 \ldots r$, which makes $1 + v'$ symmetric, we are led to the solution

$$1 + v' = [1 + B_1]^{\frac{1}{2}},$$

the right hand side being defined by its power series about the matrix $B_1 = 0$. This series converges for sufficiently small η and leads to components of v' which satisfy $|(v')_j^{\mu}| \leqslant \eta\,(1 - r\eta)^{-1}$, $j = 1, \ldots r$.

To be sure that the lower right hand corner problem has a solution in terms of vectors with only $4 - r$ components, we have to be sure that the rank of B_2 is $\leqslant 4 - r$. This follows immediately by an argument which we used several times in the proof of Lemma 2. The right hand side of (47) is a matrix of rank $\leqslant 4$. The $r \times r$ matrix $1 + B_1$ is of rank r, therefore B_2 is of rank $\leqslant 4 - r$. (Compute the determinant of all principal minors of the right hand side of (47), which have $1 + B_1$ in their upper left hand corner. They vanish if they have more than four rows and columns.) Then, by the theorem quoted in (39), we know v_k', $k = r+1, \dots n$, exist such that $v_k' \cdot v_l' = (B_2)_{kl}$, $k, l = r+1, \dots n$. Of course, there is a variety of sets of vectors v_k', $k = r+1, \dots n$ satisfying this last relation. We have to be sure that sets can be chosen so that their components are uniformly small when the matrix elements of B_2 are small. It can be shown that the v'_k can always be chosen so that*

$$|v_k'^{\mu}| \leqslant 8^{\frac{1}{2}[(4-r)-1]} [\sup_{i,j} |(B_2)_{ij}|]^{\frac{1}{2}}. \tag{48}$$

Collecting the estimates of the v_k' and $|(B_2)_{ij}|$ we see that, when η is sufficiently small, components of the v_i satisfying (45) can always be chosen so as to satisfy (44).

Lemmas 1 and 2 enabled us to prove that an invariant analytic function, f, (satisfying the hypotheses of the theorem) is necessarily a single-valued function on $\mathfrak{M}_n$. The continuity of f on $\mathfrak{M}_n$ is an immediate consequence of Lemma 3, because it demonstrates that small neighbourhoods of any point, P, on $\mathfrak{M}_n$ can have pre-images in the space of vectors which are small neighbourhoods of a pre-image of P.

We now turn to the proof of the analyticity of f on $\mathfrak{M}_n$.

Lemma 4.

If $f(z_1, \dots z_n)$ is a function of the vector variables $z_1, \dots z_n$ analytic in the tube (extended tube) and invariant under transformations of $L^{\uparrow}$, then the following equations are satisfied at every point of the tube (extended tube).

$$\sum_{j=1}^{n} \left(z_{j\mu} \frac{\partial f}{\partial z_j^{\nu}} - z_{j\nu} \frac{\partial f}{\partial z_j^{\mu}} \right) = 0. \tag{49}$$

Proof.

Let $\Lambda(a)$, $-\infty < a < \infty$, be any one parameter subgroup of real Lorentz transformations, and $z_1, \dots z_n$ a point of the tube (extended tube). Differentiating the identity

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* The inequality (48) can be proved by going through the classical induction proof of (39) estimating the size of each term. We are indebted to V. Bargmann for pointing out (48), as well as showing us a version of the proof of this Lemma which we have followed rather closely.

$$f(\Lambda(a)z_1, \ldots \Lambda(a)z_n) = f(z_1, \ldots z_n)$$

with respect to a, we obtain

$$\frac{df}{da} = \sum_{j=1}^{n} \frac{\partial f}{\partial(\Lambda(a)z_j)^{\mu}} \frac{\partial(\Lambda(a)z_j)^{\mu}}{\partial a} = 0.$$

At $a = 0$, we have

$$\left.\frac{\partial(\Lambda(a)z_j)^{\mu}}{\partial a}\right|_{a=0} = \lambda^{\mu}{}_{\nu}\, z_j{}^{\nu},$$

where $\lambda^{\mu}{}_{\nu}$ is a real 4×4 matrix satisfying

$$\lambda^{\mu}{}_{\nu} = -\lambda_{\nu}{}^{\mu}, \tag{50}$$

and defining the one parameter subgroup. Hence

$$\sum_{j=1}^{n} \lambda^{\mu}{}_{\nu}\, z_j{}^{\nu} \frac{\partial f}{\partial z_j{}^{\mu}} = 0. \tag{51}$$

Now, any real 4×4 matrix satisfying (50) generates a one parameter subgroup, so we may take λ to have zero matrix elements except for a fixed pair $\lambda_{\mu\nu}$ and $\lambda_{\nu\mu}$, then (51) reduces to (49) and the Lemma is proved.

Considered for fixed z_k, the equations (49) are a set of linear equations in the $4\,n$ unknowns $\frac{\partial f}{\partial z_j{}^{\mu}}$, $j = 1, \ldots n$, $\mu = 0, 1, 2, 3$. There are at most 6 independent equations. The derivatives of any invariant function must satisfy this set of equations at each point of the tube (extended tube). Of course, the coefficient matrix of the equations varies from point to point.

Lemma 5.

If at a point, $z_1, \ldots z_n$, of the extended tube, the number of linearly independent solutions of the equations (49) is ϱ and this ϱ dimensional linear manifold of solutions is spanned by the solutions which come from ϱ scalar products, then any invariant analytic function, f, may be represented in a neighbourhood of $z_1, \ldots z_n$ of the extended tube as a convergent power series in the ϱ scalar products, i.e., f is an analytic function of the ϱ scalar products at $z_1, \ldots z_n$.

Proof.

In the customary nomenclature, a set of analytic functions $f^{(i)}$ $i = 1, \ldots m \leq 4\,n$ of the four-vector variables $z_1, \ldots z_n$ is functionally independent at the point $z_1, \dots z_n$ if the Jacobian matrix $\dfrac{\partial f^{(i)}}{\partial z_j{}^{\mu}}$ ($i = 1, \dots m$ labels rows; $j = 1, \dots n$, $\mu = 0, 1, 2, 3$ label columns) has rank m at $z_1, \dots z_n$. In other words, the m rows of the Jacobian matrix regarded as $4\,n$ component vectors with components labeled by j and μ are linearly independent. Since functional independence at a point is equivalent to the requirement that the determinant of some $m \times m$ minor of the Jacobian matrix be different from zero, functional independence at a point implies functional independence in some neighbourhood of the point.

In this terminology, the hypothesis of the Lemma is that ϱ functionally independent scalar products exist. We shall denote these scalar products by $f^{(i)}$, $i = 1, \dots \varrho$. Since the Jacobian matrix has rank ϱ, there exists a ϱ element subset, T, of the $4\,n$ variables $z_j{}^{\mu}$, $j = 1, \dots n$, $\mu = 0, 1, 2, 3$, such that the determinant of the square matrix

$$\frac{\partial f^{(i)}}{\partial z_j{}^{\mu}} \quad i = 1, \dots \varrho,\ z_j{}^{\mu} \,\varepsilon\, T$$

is non-zero.

Then, by the implicit function theorem for several complex variables,* the $z_j{}^{\mu} \varepsilon\, T$ are analytic functions of $f^{(i)}$ $i = 1, \dots \varrho$ in a neighbourhood of $f^{(i)}(z_1, \dots z_n)$ and

$$f(z_1, \dots z_n) = g\,(f^{(1)}, \dots f^{(\varrho)}, h^{(1)}, \dots h^{(4n-\varrho)}),$$

where g is analytic in the variables $f^{(i)}$, $i = 1, \dots \varrho$ and $h^{(j)}$, $j = 1, \dots 4\,n - \varrho$ in a neighbourhood of

$$f^{(1)}(z_1, \dots z_n), \dots f^{(\varrho)}(z_1, \dots z_n),\ h^{(1)}(z_1, \dots z_n), \dots h^{(4n-\varrho)}(z_1, \dots z_n).$$

The variables $h^{(j)}$ are the $z_k{}^{\mu}$ which are not in T. The variables

$$f^{(i)},\ i = 1, \dots \varrho \text{ and } h^{(j)},\ j = 1, \dots 4\,n - \varrho$$

are obviously functionally independent in a neighbourhood of $z_1, \dots z_n$.

The derivatives of f can now be expressed in terms of the derivatives of g as follows:

$$\frac{\partial f}{\partial z_j{}^{\mu}} = \sum_{i=1}^{\varrho} \frac{\partial g}{\partial f^{(i)}} \frac{\partial f^{(i)}}{\partial z_j{}^{\mu}} + \sum_{k=1}^{4n-\varrho} \frac{\partial g}{\partial h^{(k)}} \frac{\partial h^{(k)}}{\partial z_j{}^{\mu}}.$$

However, according to the hypothesis of the Lemma, at the point $z_1, \dots z_n$

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* See B and M, p. 39, theorem 9.

$$\frac{\partial f}{\partial z_j^{\mu}} = \sum_{i=1}^{\varrho} \alpha_i \frac{\partial f^{(i)}}{\partial z_j^{\mu}}.$$

Hence,

$$\sum_{i=1}^{\varrho} \left(\frac{\partial g}{\partial f^{(i)}} - \alpha_i \right) \frac{\partial f^{(i)}}{\partial z_j^{\mu}} + \sum_{i=1}^{4n-\varrho} \frac{\partial g}{\partial h^{(i)}} \frac{\partial h^{(i)}}{\partial z_j^{\mu}} = 0.$$

But regarded as vectors with $4n$ components, $\frac{\partial f^{(i)}}{\partial z_j^{\mu}}$, $i = 1, \ldots \varrho$ and $\frac{\partial h^{(i)}}{\partial z_j^{\mu}}$, $i = 1, \ldots 4n - \varrho$ are linearly independent. (That is what the functional independence of the $f^{(i)}$ and $h^{(i)}$ means.) Therefore, their coefficients in this equation must vanish. In particular,

$$\frac{\partial g}{\partial h^{(i)}} = 0, \; i = 1, \ldots 4n - \varrho,$$

so g is independent of the $h^{(i)}$ and the Lemma is proved.

Lemma 6.

Let N be the maximum number of linearly independent vectors contained in the set of 4-vectors $z_1, \ldots z_n$, then the set of six linear equations

$$\sum_{j=1}^{n} (z_{j\mu} X_{j\nu} - z_{j\nu} X_{j\mu}) = 0 \quad \mu, \nu = 0, 1, 2, \quad 3\ \mu < \nu \tag{52}$$

for the $4n$ quantities $X_{j\mu}$, $j = 1, \ldots n$, $\mu = 0, 1, 2, 3$ has the rank

N	1	2	$\geqslant 3$
rank	3	5	6

Proof.

If R is a non-singular linear transformation of four dimensional space, it is clear that the set of equations (52) has the same rank as the set

$$\sum_{j=1}^{n} [(Rz_j)_{\mu} (RX_j)_{\nu} - (Rz_j)_{\nu} (RX_j)_{\mu}] = 0, \quad \mu_1 v = 0, 1, 2, 3$$

for the $4n$ quantities $(RX_j)_{\mu}$. Then, with a suitable choice of R, the last $(4 - N)$ components of vectors Rz_j, $j = 1, \ldots n$ can be made to vanish. Having reduced the problem to this simplified form, we drop the R and assume the last $4 - N$ of the components of the Z_j vanish.

To find the number of linearly independent equations (52), consider a possible linear dependence among them

$$\sum_{\substack{\mu,j=0\\ \mu<\nu}}^{3} a^{\mu\nu} \sum_{j=1}^{n} (z_{j\mu}X_{j\nu} - z_{j\nu}X_{j\mu}) = 0$$

for all $X_{l,k}$, i.e.,

$$\sum_{\substack{\mu=0\\ \mu<k}}^{3} a^{\mu\varkappa} z_{l\mu} - \sum_{\substack{\nu=0\\ k<\nu}}^{3} a^{\varkappa\nu} z_{l\nu} = 0\,,\ l = 1, \ldots n\,,\ \varkappa = 0, 1, 2, 3$$

or

$$Az_l = 0\,,\ l = 1, \ldots n,$$

where we have written

$$-A^{\nu\mu} = A^{\mu\nu} = a^{\mu\nu},\ \mu<\nu,\ A^{\mu\mu} = 0,\ \mu, \nu = 0, 1, 2, 3.$$

Thus, the rank of the equations (52) is six minus the number of linearly independent skew symmetric matrices, A such that $Az_i = 0$, $i = 1, 2, \ldots n$. The most general A has the form $\begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix}$ where A' is an arbitrary anti-symmetric $(4-N)\times(4-N)$ matrix. The numbers tabulated in the Lemma are just six minus the number of linearly independent A'.

Lemma 7.

Let $z_1, \ldots z_n$ be a set of n four-vectors of which some four-element subset (or n element subset if $n<4$) is linearly independent. Then, the number of linearly independent solutions of equations (52) which arise from scalar products $f^{(i)}$ of the four vectors according to

$$X_{j\nu}^{(i)} = \frac{\partial f^{(i)}}{\partial z_{j\nu}}$$

is $4\,n-6$ if $n \geqslant 3$, is 3 if $n = 2$, and 1 if $n = 1$.

Proof.

It is clear that, for $n>3$, *not more* than $4\,n-6$ linearly independent solutions of equations (52) can be obtained from scalar products because there are at most $4\,n-6$ functionally independent scalar products. This follows immediately from equation (34), which expresses $z_k \cdot z_l$, k, $l \geqslant 5$ in terms of $z_i \cdot z_j$, where $i = 1, 2, \ldots n, j = 1, 2, 3, 4$. Of these $4\,n$, 6, namely $z_i \cdot z_j$, $i>j$, i = 1, 2, 3, 4 are expressible in terms of the rest. For $n = 3, 2, 1$ it is obvious that there are respectively 6, 3 and 1 independent scalar products at most and therefore 6, 3, and 1 linearly independent solutions of (52) at most.

To show that these upper limits on the number of solutions are actually realized under the hypothesis of the Lemma, we proceed as follows.

Denote a vector in the space of solutions of (52) by $(\zeta_1, \ldots \zeta_n)$ where the ζ_i are four-dimensional vectors. This notation is chosen so that an invariant analytic function F generates a solution $\left(\frac{\partial F}{\partial z_1} \ldots \frac{\partial F}{\partial z_n}\right)$ where $\frac{\partial F}{\partial z_j}$ stands for the four-dimensional vector with components, $\frac{\partial F}{\partial z_j^{\mu}}$. In this notation, the solution of (52) which comes from the scalar product $z_j \cdot z_k$ is

$$(\underset{\xrightarrow{k^{th}\ \text{place}}}{\underset{\xrightarrow[j^{th}\ \text{place}]{}}{0, \ldots 0, z_k}, 0, \ldots 0, z_j}, 0, \ldots 0), \tag{53}$$

and from z_j^2

$$(\underset{\xrightarrow[j^{th}\ \text{place}]{}}{0, \ldots 0, 2\, z_j}, 0, \ldots 0), \tag{54}$$

By convention, we will accept (54) as the value of (53) for $j = k$.

The most general solution of (52) arising from scalar products is of the form

$$(\zeta_1, \ldots \zeta_n) = \sum_{j,k=1}^{n} a_{jk}\,(0, \ldots 0, z_k, 0, \ldots 0, z_j, 0, \ldots 0).$$

Here, evidently

$$\zeta_j = \sum_{k=1}^{n} (a_{jk} + a_{kj})\, z_k,$$

so that the antisymmetric part of the matrix a_{ik} does not contribute and a_{jk} may as well be chosen symmetric.

Now, for convenience, let the first four z_j (or first n if $n < 4$) be the linearly independent set whose existence is assumed in the Lemma. Then we can write

$$z_k = \sum_{l=1}^{m} b_{kl} z_l,\ k = 1, \ldots n, \tag{55}$$

where $m = \min(4, n)$ and b is an $n \times m$ matrix of rank m. Substituting (55) into the expression for the solution vector we see that, in the basis for solution vectors provided by $(z_{l_1}, z_{l_2}, \ldots z_{l_n})$, $l_j = 1, \ldots m$, $j = 1, \ldots n$, the most general solution arising from scalar products is of the form of an $n \times m$ matrix AB, where A is an arbitrary symmetric $n \times n$ matrix and B is a fixed $n \times m$ matrix of rank m.

To count the number of linearly independent AB simply, write B as the product of a non-singular $n \times n$ matrix B' and the $n \times m$ matrix whose first m rows form the unit matrix and whose last $n - m$ are zero:

$$B = B'\begin{pmatrix}1\\ \hline 0\end{pmatrix}.$$

This is always possible because B is of rank m. The number of linearly independent matrices AB is the same as the number of linearly independent matrices

$$(B')^{\mathsf{T}}AB'\begin{pmatrix}1\\ \hline 0\end{pmatrix},$$

i.e., the same as the number of linearly independent matrices of the form

$$\begin{pmatrix}S_1\\ S_2\end{pmatrix},$$

where S_2 is an arbitrary $(n-m)\times m$ matrix and S_1 is an arbitrary symmetric $m\times m$ matrix. There are obviously $m(n-m)+\frac{1}{2}\,m(m+1)$ linearly independent of these, which immediately yields the statement of the Lemma.

Completion of the proof.

Lemmas 1 to 7 establish the single valuedness, boundedness, and continuity of f everywhere on $\mathfrak{M}_n$, and its analyticity on $\mathfrak{M}_n$ at every nonexceptional point, i.e., every point where the matrix $z_i\cdot z_j$, $i,j = 1, 2 \ldots n$, has the maximum possible rank, min $(4,n)$. To complete the proof of the theorem, we want to show that in those cases where the set of exceptional points is not singular in the sense of algebraic geometry, viz. $n = 1, 2, 3, 4$, f is also analytic there. For this purpose, we use a standard theorem on removable singularities which asserts*: Let f be a function which is analytic in a neighbourhood of a point, P, with the possible exception of a variety passing through P, the variety being defined as the set of zeros of a function analytic in the neighbourhood of P. Suppose that f is continuous or merely bounded throughout the neighbourhood of P. Then f is analytic throughout the neighbourhood of P. In our case, the variety is obtained by setting the analytic function det $(z_j\cdot z_k) = 0$. The required analyticity and continuity of f having been established by our Lemmas 1–7, the proof of the theorem is complete.

2. The varieties $\mathfrak{M}_n$.

As we have seen in Lemma 3, every rank $\leqslant 4$ complex symmetric matrix is a matrix of scalar products of four vectors, so that $\mathfrak{M}_n$ is a subset of the set of all complex symmetric $n\times n$ matrices of rank $\leqslant 4$. The same Lemma shows that $\mathfrak{M}_n$ is an open subset. It is clearly connected because it is the continuous image of a connected set, the tube. It is also simply connected by virtue of Lemma 3, although we shall forgo a formal proof. (The idea is, given a closed curve on $\mathfrak{M}_n$ which has to be shrunk to a point, to construct a closed curve of vectors in the tube whose image in $\mathfrak{M}_n$ is the given curve. Then, because the tube is simply connected, the curve of vectors can be shrunk to a point which implies that their image can be shrunk to a point.) Not every rank < 4 complex symmetric $n \times n$ matrix, Z, is in $\mathfrak{M}_n$, for example if Z has real positive diagonal elements it is not in $\mathfrak{M}_n$. We shall not attempt a quantitative characterization of $\mathfrak{M}_n$ at this stage, but only remark that it need not be the natural domain of analyticity for the analytic functions which occur in field theory. For example, the first named author showed in his thesis[9] that the local commutativity conditions, I equation (11), always make it possible to extend the analytic function determined by the threefold vacuum expectation value $(\Psi_0 \varphi(x_1) \varphi(x_2) \varphi(x_3) \Psi_0)$ beyond $\mathfrak{M}_2$. On the other hand, it is clear that such functions cannot in general be extended to all complex symmetric rank $\leqslant 4$, $n \times n$ matrices because they must have branch lines in order to conform with physical requirements. (See, for example, the discussion of $F^{(2)}(z_{12}^2)$ in I, Section 4.)

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* B and M, p. 173, theorem 5.

The restriction to $n \times n$ matrices of rank $\leqslant 4$ is of course no restriction at all for $n \leqslant 4$ so the $\mathfrak{M}_n$ for $n \leqslant 4$ are open sets in Euclidean $1/2\, n(n+1)$ space. For $n \leqslant 5$ the restriction to rank $\leqslant 4$ on an $n \times n$ matrix Z_{ij} can be stated

$$\det \begin{pmatrix} Z_{i_1 j_1} & \ldots & Z_{i_1 j_5} \\ \\ Z_{i_5 j_1} & \ldots & Z_{i_5 j_5} \end{pmatrix} = 0$$

for each pair of five element subsets $i_1, \ldots i_5$ and $j_1, \ldots j_5$ of $1, 2, \ldots n$. The tangent spaces of $\mathfrak{M}_n$ are determined by taking the differential of the left hand side of (56). The result is a set of linear equations for the dZ_{ij} whose coefficients are determinants of 4×4 minors of Z. At any point of $\mathfrak{M}_n$ where all determinants of 4×4 principal minors of Z vanish, these equations are satisfied for any choice of dZ_{ij}. Consequently, at such a singular point, the tangent space is $1/2\, n\, (n+1)$ dimensional. On the other hand, as we learned in Lemma 7, the dimension at a non-singular point is $4n-6$.

Now we want to study the relation of the points of $\mathfrak{M}_n$ to those of the form $Z_{ij} = \xi_i \cdot \xi_j$, ξ_i real, $i,j = 1, \ldots n$. We will refer to such Z as physical because the arguments of the physically given vacuum expectation values are real vectors. We remark that every physical point Z is either in the interior of $\mathfrak{M}_n$ or on its boundary, because $\xi_j \cdot \xi_k$ is the limit of $(\xi_j - i\eta_j) \cdot (\xi_k - i\eta_k)$ as the η's approach zero. No physical point Z can be in the interior of $\mathfrak{M}_n$ if any of the vectors $\xi_j, j = 1, 2, \ldots, n$ (of which Z is the set of scalar products) is light-like or time-like. To see this, consider a general point $\xi'_j - i\eta_j$, $j = 1, 2, \ldots n$ of the tube. The corresponding point of $\mathfrak{M}_n$ is given by

$$Z'_{jk} = \xi'_j \cdot \xi'_k - \eta_j \cdot \eta_k - i(\xi'_j \cdot \eta_k + \eta_j \cdot \xi'_k).$$

If Z' is to be real, it is necessary that each of the ξ'_j be space-like since it is orthogonal to a vector inside the light cone. But then the diagonal elements of Z', $\xi'^2_j - \eta^2_j, j = 1, \ldots n$, are negative so that Z' can be a physical point Z, $Z_{jk} = \xi_j \cdot \xi_k$, $j, \varkappa = 1, \ldots n$ only if the vectors ξ_j satisfy $\xi^2_j = \xi'^2_j - \eta^2_j$. On the other hand, as we now will show, some physical points with space-like ξ_j do lie in the interior of $\mathfrak{M}_n$. Since this is a fact of considerable physical significance, and the geometrical relationships are rather involved, it is worth introducing some notation to describe the situation. We denote by S_n the set of all physical points, Z, which arise from space like vectors, i.e. of the form $Z_{jk} = \xi_j \cdot \xi_k$, $j, k = 1, \ldots n$ with ξ_j real and space-like. The subset of S_n which arises from ξ_j, $j = 1, \ldots n$, lying in a space-like three dimensional linear manifold, we will denote by T_n. We will also call T_n the equal time-manifold since it is the set of matrices whose elements can be taken as scalar products of vectors arising from vectors $\xi_j = x_j - x_{j+1}$, $j = 1, \ldots n$, where the x_j have equal first components.

We first prove that a subset of T_n lies in the interior of $\mathfrak{M}_n$, and then pass to neighbourhoods of that subset. Consider the vectors $\xi'_j - i\eta_j$, $j = 1, \ldots n$, where $\eta_j = \alpha_j \eta$, α_j is a real positive number, η is a real unit vector in the direction of the time axis, and ξ'_j, $j = 1, \ldots n$ are real vectors with zero component in the time direction and in one space direction, say the direction of the third axis. Then,

$$Z' = (\xi'_j - i\eta_j) \cdot (\xi'_k - i\eta_k) = \xi'_j \cdot \xi'_k - \alpha_j \alpha_k = \xi_j \cdot \xi_k,$$

where ξ_j is defined as ξ'_j plus a vector along the third axis with component α_j. The vectors ξ_j evidently all have zero time components, so that Z' is in T_n. Although the point $\xi_1, \ldots \xi_n$ does not lie in the tube, it must, by Lemmas 2 and 3, lie in the extended tube. Furthermore, by suitable choice of the components of ξ'_j, $j = 1, \ldots n$, it can be arranged that Z' has rank three. By Lemma 3, it then follows that vectors lying in neighbourhoods of ξ_j have scalar products which cover full neighbourhoods of Z' in $\mathfrak{M}_n$. Thus, the fact that the particular points Z' chosen above lie in $\mathfrak{M}_n$ implies that the physical points which arise from all $\xi_1, \ldots \xi_n$ lying in a suitably small neighbourhood of the chosen $\xi_1, \dots \xi_n$ also lie in $\mathfrak{M}_n$. This shows that $T_n \cap \mathfrak{M}_n$ and $S_n \cap \mathfrak{M}_n$ have the same dimension as T_n and S_n, respectively.

Not all of S_n lies in $\mathfrak{M}_n$, but we will not attempt to prove this now nor to characterize those points of S_n which lie on the boundary of $\mathfrak{M}_n$. We content ourselves here with the simplest consequences of the preceding results on T_n and S_n. The set $S_n \cap \mathfrak{M}_n$ has been shown to be of the same dimension as S_n and to contain, for suitably chosen ε, all real symmetric matrices Z' of rank $\leqslant 4$ satisfying $|Z'_{jk} - \xi_j \cdot \xi_k| < \varepsilon$. This set is a real environment* for an analytic function defined on $\mathfrak{M}_n$; an analytic function f is uniquely determined all over $\mathfrak{M}_n$ if its values are given on this set. We shall see in the next section that this result has important physical consequences.

The points of T_n are always of rank $\leqslant 3$ so that for $n \geqslant 5$, $T_n \cap \mathfrak{M}_n$ lies in the singular subset of $\mathfrak{M}_n$ where the ordinary definition of analyticity fails. The set $T_4 \cap \mathfrak{M}_4$ is of (real) dimension nine while $S_4 \cap \mathfrak{M}_4$ is of dimension ten so that $T_4 \cap \mathfrak{M}_4$ is not a real environment. On the other hand, $T_1 \cap \mathfrak{M}_1$, $T_2 \cap \mathfrak{M}_2$ and $T_3 \cap \mathfrak{M}_3$ have the same dimension as $S_1 \cap \mathfrak{M}_1$, $S_2 \cap \mathfrak{M}_2$ and $S_3 \cap \mathfrak{M}_3$, so that they form real environments for analytic functions on $\mathfrak{M}_1$, $\mathfrak{M}_2$, and $\mathfrak{M}_3$, respectively.

3. Physical Applications.

Some physical applications of the theorem of Section 1 were already discussed in I (See, for example, the formulation of local commutativity given in I equation (11).) They arise, like those to be discussed below, when the theorem is applied to the invariant analytic function

$$F^{(n)}(z_1, z_2, \dots z_{n-1})$$

whose boundary value, as all $\eta_j \to 0$, $j = 1, 2, \dots n$, is the vacuum expectation value

$$F^{(n)}(\xi_1, \dots \xi_{n-1}) = (\Psi_0, \varphi(x_1) \dots \varphi(x_n) \Psi_0).$$

Here, in a notation somewhat different from I, we have written

$$z_j = \xi_j - i\eta_j \text{ and } \xi_j = x_j - x_{j+1} \quad j = 1, \dots n-1.$$

The first consequence of the theorem is that $F^{(n)}(\xi_1, \dots \xi_{n-1})$ *is an analytic function of the real variables* $Z_{ij} = \xi_i \cdot \xi_j$ $i, j = 1, \dots n-1$ *in an open subset of the set where all* ξ_j *are space-like,* i.e. *in the notation of the preceding section, as long as* Z *belongs to a certain open subset of* $S_{n-1} \cap \mathfrak{M}_{n-1}$. This conclusion follows immediately from the analysis which showed that $S_{n-1} \cap \mathfrak{M}_{n-1}$ is a real environment in $\mathfrak{M}_{n-1}$. (Recall that $\mathfrak{M}_n$ is the set of all complex symmetric matrices of the form $z_i \cdot z_j$, $i,j = 1, 2, \ldots n$, with z_i in the tube. Of course, on $\mathfrak{M}_n$ for $n \geqslant 5$, the ordinary definition of analyticity has no meaning at the exceptional points where the rank of $z_i \cdot z_j$ is less than 4. See the discussion in the outline of the proof in Section 1.

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* See B and M, pp. 33–34, for the definition of a real environment in Euclidean space. The same definition works here because a neighbourhood of a non-singular point in $\mathfrak{M}_n$ is essentially a Euclidean neighbourhood.

Furthermore, *the vacuum expectation value*

$$(\Psi_0, \varphi(x_1) \ldots \varphi(x_n) \Psi_0)$$

is uniquely determined from its values for space-like separated $x_1, \ldots x_n$. It is possible to regard this result as a quantitative formulation of the intuitive feeling that in a Lorentz invariant theory the equivalence of descriptions in different Lorentz frames should somehow restrict the possible correlations between the values of physical quantities at different points in space time.

For $F^{(2)}(\xi_1)$, $F^{(3)}(\xi_1, \xi_2)$, and $F^{(4)}(\xi_1, \xi_2, \xi_3)$ an even more striking result holds:

$$(\Psi_0, \varphi(x_1)\varphi(x_2)\Psi_0), \quad (\Psi_0, \varphi(x_1)\varphi(x_2)\varphi(x_3)\Psi_0),$$

and

$$(\Psi_0, \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\Psi_0)$$

are uniquely determined from their values at equal times, i.e., in the notation of the preceding section from their values for $Z_{ij} = \xi_i \cdot \xi_j$ *with* $Z \varepsilon T_1$, T_2, T_3, *respectively*. We want to emphasize that all three of these results hold in both local and non-local field theory.

The most important application of the preceding remarks we know of is to the proof of the following theorems which are extensions of results stated by R. HAAG[10].

Theorem (Generalized HAAG's Theorem First Part).

Let two theories (distinguished by a subscript $j = 1, 2$) of a neutral scalar field be given whose canonical variables are related at time t by a unitary transformation, V:

$$U_j(\vec{a}, R)\varphi_j(\vec{x}, t)U_j(\vec{a}, R)^{-1} = \varphi_j(R\vec{x}+\vec{a}, t) \quad j = 1, 2 \tag{57}$$

$$U_j(\vec{a}, R)\pi_j(\vec{x}, t)U_j(\vec{a}, R)^{-1} = \pi_j(R\vec{x}+\vec{a}, t) \quad j = 1, 2. \tag{58}$$

(Transformation law of field variable and canonical conjugate under Euclidean transformation.)

$$\left.\begin{aligned} &[\pi_j(\vec{x}, t), \varphi_j(\vec{y}, t)] = i^{-1}\delta(\vec{x}-\vec{y}) \quad j = 1, 2 \\ &[\pi_j(\vec{x}, t), \pi_j(\vec{y}, t)] = 0, \ [\varphi_j(\vec{x}, t), \varphi_j(\vec{y}, t)] = 0 \end{aligned}\right\} \tag{59}$$

(Commutation Relations)

$$\varphi_2(\vec{x}, t) = V\varphi_1(\vec{x}, t)V^{-1}, \quad \pi_2(\vec{x}, t) = V\pi_1(\vec{x}, t)V^{-1}. \tag{60}$$

Here $(\vec{a}, R)$ represents the Euclidean transformation: rotate by R and translate by $\vec{a}$; the unitary transformations, $U_j(\vec{a}, R)$, give the corresponding transformations of the states in the two theories.

Then

$$U_2(\vec{a}, R) = VU_1(\vec{a}, R)V^{-1}. \tag{61}$$

If each of the theories contains a unique normalizable state Ψ_{0j}, $j = 1, 2$, invariant under Euclidean transformation:

$$U_j(\vec{a}, R)\Psi_{0j} = \Psi_{0j} \quad j = 1, 2,$$

then

$$c\,\Psi_{02} = V\Psi_{01},$$

where c is a constant of absolute value 1.

Proof.

From (57), (58), and (60) we can easily derive that the operators

$$U_1(\vec{a}, R)^{-1}V^{-1}U_2(\vec{a}, R)V \tag{62}$$

commute with $\varphi_1(\vec{x}, t)$ and $\pi_1(\vec{x}, t)$ for all $\vec{x}$. Because the φ_1 and π_1 form an irreducible set* of operators, (62) must be a constant multiple of the identity operator: $\omega(\vec{a}, R)\,1$ and

$$U_2(\vec{a}, R) = \omega(\vec{a}, R)VU_1(\vec{a}, R)V^{-1}. \tag{63}$$

From (63), it follows that $(\vec{a}, R) \to \omega(\vec{a}, R)$ is a continuous unitary one dimensional representation of the Euclidean group and therefore $\omega(\vec{a}, R) = 1$.[11] This completes the proof of the first half of the theorem. To prove the second half, note that

$$U_1(\vec{a}, R)\Psi_{01} = \Psi_{01}$$

and (63) imply

$$U_2(\vec{a}, R)V\Psi_{01} = V\Psi_{01}.$$

Thus, by the uniqueness of Ψ_{02}, $V\Psi_{01}$ is a multiple, c, of Ψ_{02}. Since V is unitary, $|c| = 1$.

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* That $\varphi_1(\vec{x}, t)$ and $\pi_1(\vec{x}, t)$ form an irreducible set is what we mean by our assumption that the theory is a theory of the scalar field φ_1. This assumption is made for simplicity. In a theory in which the field φ_1 interacted with a spinor field, ψ, one would only have to introduce the hypothesis that $\varphi_1, \pi_1, \psi, \bar{\psi}$, form an irreducible set, together with the appropriate extension of (57)...(61), to obtain an analogous theorem.

It should be noticed that only the properties of the fields and states under Euclidean transformation at time t have been used in the proof of the theorem. To one accustomed to the formalism of non-relativistic quantum mechanics, the conclusion of the theorem is in no way surprising; V always exists in such theories. Of course, there, V is a function of time, and physically different theories will give a different time dependence for V. The surprise comes when, following HAAG, one combines the preceding assumptions with those of relativistic invariance.

Theorem (Generalized HAAG's Theorem Part II).

Let two theories of a neutral scalar field be given satisfying the hypotheses of the preceding theorem. Let the theories be invariant under inhomogeneous Lorentz transformations (a,Λ) and suppose the fields transform as follows:

$$U_j(a,\Lambda)\,\varphi_j(x)\,U_j(a,\Lambda)^{-1} = \varphi_j(\Lambda x+a) \qquad j = 1, 2. \tag{64}$$

Suppose further that the states Ψ_{0j} are invariant under inhomogeneous Lorentz transformation

$$U_j(a,\Lambda)\,\Psi_{0j} = \Psi_{0j} \qquad j = 1, 2, \tag{65}$$

and that no states of negative energy exist.

Then the first four vacuum expectation values are equal in the two theories.

$$(\Psi_{01}, \varphi_1(x_1)\dots\varphi_1(x_n)\,\Psi_{01}) = (\Psi_{02}, \varphi_2(x_1)\dots\varphi_2(x_n)\,\Psi_{02}) \tag{66}$$

Proof.

From the preceding theorem we have for equal times $x_1^0 = x_2^0 = \dots = x_n^0$:

$$\begin{aligned}(\Psi_{01}, \varphi_1(x_1)\dots\varphi_1(x_n)\,\Psi_{01}) &= (V\Psi_{01}, V\varphi_1(x_1)V\dots.V\varphi_1(x_n)V^{-1}V\Psi_{01})\\ &= (\Psi_{02}, \varphi_2(x_1)\dots\varphi_2(x_n)\,\Psi_{02}).\end{aligned}$$

Thus, all vacuum expectation values are equal for equal times in the two theories. For $n = 1, 2, 3, 4$, equality for all times $x_1^0 \neq x_2^0 \neq \dots \neq x_n^0$ follows from equality for equal times by the argument presented earlier in this section. This completes the proof. The hypotheses about the absence of negative energy states and the existence of the vacuum are necessary in order that the vacuum expectation values be boundary values of analytic functions to which our previous analysis applies.

It should be noticed that we have *not* made the assumption that the two theories transform according to equivalent representations of the inhomogeneous Lorentz group; our hypotheses do not exclude a priori the possibility that the two theories have different bound states, for example. Further, we have not assumed any particular transformation law for the operators π under Lorentz transformations and time translations. Only the behaviour of $\pi(\vec{x}, t)$ for one particular time under Euclidean transformations is needed.

The uniqueness of the vacuum state is crucial to the argument. If it were possible to form normalizable states of zero three-momentum from states of mass greater than zero, the hypothesis of a unique normalizable state of zero three-momentum would be unnatural and the second theorem physically trivial. V could then carry Ψ_{01} into a superposition of Ψ_{02} and those other states of zero three-momentum. However, Wigner's analysis of the unitary representations of the inhomogeneous Lorentz group[12] shows that states can never be normalizable which are superpositions of states of mass greater than zero and have zero three-momentum, and Haag's theorem is very far from physically trivial.

As a particular case one can take the field φ_1 to be a free field satisfying

$$(\Box + m^2)\, \varphi_1(x) = 0\,, \;\; [\varphi_1(x),\, \varphi_1(y)] = i^{-1} \Delta(x-y).$$

Then we conclude: *no theory of interaction exists in which the ordinary representation of the annihilation and creation operators is used and the first four vacuum expectation values differ from their free field values. If relativistic theories of interaction exist with vacuum expectation values, $F^{(n)}$, different from the free field values for $n = 1, 2, 3, 4$, either they must use other representations of the canonical commutation relations or they do not satisfy the canonical commutation relations at all.* (This is essentially HAAG's conclusion[10].) This result shows that the situation which was found by WIGHTMAN and SCHWEBER[13] in a special non-relativistic example is typical of relativistic theories of interaction which satisfy the canonical commutation relations (if such exist at all): For each different value of the coupling constant one must use an inequivalent representation of the commutation relations (assuming that different values of the coupling constant will give rise to some difference in the vacuum expectation values $F^{(n)}$ for $n = 1, 2, 3, 4$.) Of course, the converse is not true; inequivalent representations of the commutation relations need not always give rise to physically distinct theories.

From both the aesthetic and physical point of view, the version of the generalized Haag's theorem proved here is somewhat deficient because it only asserts the equality of the first four vacuum expectation values. It seems physically plausible that two theories in which the two-particle propagator, the vertex part, and the two-particle scattering for all energies are identical (as they must be if the first four vacuum expectation values are identical) should be completely identical. On this basis, one would conjecture the aesthetically more satisfying result that *all* vacuum expectation values coincide, which would (from the work of I Section 5) indeed imply the physical equivalence of the two theories. To prove this result along the lines of the present paper would require one to establish a unique analytic continuation out of the equal time-manifold T_n into $\mathfrak{M}_n$; it would require an analysis going essentially beyond what we have presented in Section 1. (In fact, it is not difficult to see that under the hypothesis of Section 1, the analytic continuation is *not* unique for $n \geqslant 4$.)

A second matter which the present paper leaves untouched is the question of the existence of theories which use representations of the commutation relations different from those of a free field. If it turns out that no such representation gives rise to a relativistically invariant theory which is physically interesting, that would be very strong evidence of the incompatibility of the canonical commutation relations, relativistic invariance and interaction. In fact, it would show that the fact that a field strength renormalization constant is infinite in quantum electrodynamics[14] is not a special consequence of the Hamiltonian of the theory, but a general result arising from relativistic invariance.

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References.

1. A. S. Wightman, Phys. Rev. **101**, 860 (1956). This paper will be referred to as I. We adopt the notation of I, except when it is explicitly stated otherwise.
2. See, for example, S. Lefshetz, Algebraic Geometry, Princeton Press (1953), pp. 35–40.
3. S. Bochner and W. T. Martin, Annals of Math. **57**, 490 (1953).
4. For proof of these statements, see L. Pontrjagin, Topological Groups, Princeton Press (1939), Chapters VI and IX. See especially pp. 265–66.
5. For a proof of this basic property of analytic functions, see S. Bochner and W. T. Martin, Several Complex Variables, Princeton Press (1948), p. 34, theorem 4. We shall refer to this book as B and M.
6. See, for example, L. Kronecker, Journal für Reine u. Angew. Math. **72**, 152 (1870) or C. C. Mac Duffee, Ergebnisse der Math. **2**, 364 (1933).
7. The constructions of these paragraphs are given in detail, because they solve a problem in linear algebra not standard in the physical literature. They are well known in the mathematical literature. See, for example, E. Witt, Journal für Math. **176**, 31 (1936) or C. Chevalley, The Algebraic Theory of Spinors, Columbia University Press, New York (1954), pp. 8–15. The essential points are the following. Even for a complex valued scalar product such as $z\cdot w = z_\mu w^\mu$, provided it is non-degenerate (i. e. provided $z\cdot w=0$ for all vectors w implies $z=0$), it remains true that a linear manifold and its orthogonal manifold $M^\perp$ have dimensions satisfying

$$\dim M + \dim M^\perp = 4.$$

However, unlike the case with a positive scalar product, it can happen that M and $M^\perp$ have non-zero vectors in common; then M and $M^\perp$ do not together span the whole space. A necessary and sufficient condition that $M \cap M^\perp = (0)$ is that M have a basis whose Gram determinant is not zero. Then, all bases for M, and $M^\perp$ as well, have non-vanishing Gram determinant. In that case, a general vector has a unique decomposition into a sum of a vector in M and a vector in $M^\perp$. There is a simple formula for the projection of a vector v into a subspace with basis $v_1, \dots, v_m$

$$-[G(v_1\dots v_m)]^{-1}\det\left(\begin{array}{c|c} & v_1\cdot v \\ & \cdot \\ v_i\cdot v_j & \cdot \\ & \cdot \\ & v_m\cdot v \\ \hline v_1,\dots v_m & 0 \end{array}\right)$$

If a subspace has the property that every vector in it is of zero length, it is called isotropic. Isotropic subspaces can have dimensions no larger than half the dimension of the whole space. Most of these results follow directly from arguments of the type given in the preceding paragraphs of the proof. They will be used without comment in the remainder of the proof.

8. See, for example, C. C. MacDuffee, Ref. 6, p. 408, theorem 34.1.
9. D. Hall, thesis, Princeton (1956), Chapter III, unpublished.
10. R. Haag, Mat. Fys. Medd. Dan. Vid. Selsk. **29**, no. 12 (1955), especially pp. 30–32. In the opinion of the present authors, Haag's proof is, at least in part, inconclusive. We thank Dr. Haag for a correspondence on the matter. It will not escape the discerning reader of Haag's paper that, while we have generalized his results, eliminated one of his assumptions (the asymptotic condition), completed his proofs, and sharpened his conclusions, the essential physical points are Haag's.
11. This result follows from a determination of all representations of the Euclidean group implicit in theorems of G. Mackey, Annals of Math. **55**, 101 (1951). It is also not difficult to construct a simple direct proof.
12. E. P. Wigner, Annals of Math. **40**, 149 (1939).
13. A. S. Wightman and S. Schweber, Phys. Rev. **98**, 812 (1955). See especially p. 824.
14. G. Källén, Cern/T/GK3, report, unpublished.

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On the Bound State Problem in Quantum Field Theory (*).

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Summary. — A causal and invariant scalar field involving a stable bound state is investigated. A formula for the S-matrix is derived and it is shown that the bound state can be described by a local and invariant field operator. For simplicity only the case of spin zero particles and bound states is considered; however, the extension to other cases is possible.

Introduction.

For a relativistic quantum mechanical system with an energy momentum operator P_μ the one particle states are defined as eigen states of $-P_\mu^2$ with a discrete non-vanishing rest mass (1,2). In general, there may be several kinds of particles, each characterized by a certain value of rest mass, spin,charge, etc. Usually a division is made between elementary and composite particles. But it seems to be hard to define this distinction in a convincing manner. In the conventional formulation of quantum field theory each of the elementary

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(1) For a discussion of the particle aspect in quantum field theory compare R. Haag: *Dan. Mat. Fys. Medd.*, 29, 12 (1955), especially chap. I, no. 1.

(2) We exclude the case of particles with zero rest mass.

particles is described by a basic field operator whereas the composite particles (like the deuteron in the ground state, etc.) appear as the stable bound states of the system. But this definition of a composite particle depends, of course, on the formalism. The same particle which is regarded as an elementary particle in one formalism may be a composite particle in another (3,4).

In this paper we want to investigate whether the principle of microscopic causality sheds any new light on this question. We consider a model of a causal and invariant scalar field $A(x)$ describing just two kinds of particles, an elementary particle of mass m and a composite particle of mass M, both of spin zero. It will be shown that it is possible to define a field operator $B(x)$ for the composite particle (explicitly expressed in terms of the original field $A(x)$) which is in some respect analogous to the field operator of an elementary particle. The invariant operator $B(x)$ satisfies the requirement of microscopic causality, furthermore the S-matrix can be expressed by the vacuum τ-functions (5) of the field operator $A(x)$ and $B(x)$ exactly in the same way as in the case of elementary particles only. Therefore the principle of microscopic causality offers no possibility of distinguishing between elementary and composite particles. With respect to the S-matrix the elementary particles as well as the composite particles of the model are described by invariant field operators which satisfy the requirement of microscopic causality (6,7).

The formalism developed in this paper is closely related to other recent investigations. Starting from similar requirements one can derive the same results by applying the method of strong operator convergence in the form developed by HAAG (8,9). An equivalent formalism was also obtained from the recursion formulae for retarded functions recently proposed by NISHIJIMA as an axiomatic formulation of quantum field theory (10,11).

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(3) Compare E. FERMI and C. N. YANG: *Phys. Rev.*, **76**, 1739 (1949). In this paper the hypothesis that π-mesons may be composite particles is discussed.

(4) In this connection an interesting example is Gürsey's model of a theory of elementary particles (F. GÜRSEY: to be published). There, following a suggestion of Heisenberg, all particles appear as composite and no elementary particles correspond to the basic fields.

(5) The vacuum τ-functions are defined as the vacuum expectation values of multiple, time ordered, operator products.

(6) This possibility was first mentioned by N. BOGOLJUBOV: unpublished lecture notes.

(7) The S-matrix of the model considered is causal according to the definition given in H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **6**, 319 (1957).

(8) R. HAAG: *Proc. of the Lille Conference on Mathematical problems of the quantum theory of fields* (1957), in print.

(9) R. HAAG: preprint, to be published.

(10) K. NISHIJIMA: *Progr. Theor. Phys.*, **17**, 765 (1957).

(11) K. NISHIJIMA: preprint, to be published.

1. – General conditions.

We consider the model of a neutral scalar field described by a hermitian operator $A(x)$ assumed to be invariant under the inhomogeneous Lorentz group. We assume that the principle of microscopic causality

$$[A(x), A(y)] = 0 \qquad \text{for } (x-y)^2 > 0 \tag{1}$$

holds and that no negative eigenvalues appear in the energy and rest mass spectrum. The operators $A(x)$ will be supposed to form an irreducible operator ring. For simplicity we assume that there are just two non-vanishing discrete eigenvalues m^2 and M^2 of $-P_\mu^2$ and that

$$(\Omega, A(x)\Phi) \not\equiv 0 \qquad \text{if} \quad -P_\mu^2\Phi = m^2\Phi$$

but

$$\left.\begin{aligned}(\Omega, A(x)\Psi) &\equiv 0\\ (\Omega, A(x)A(y)\Psi) &\not\equiv 0\end{aligned}\right\} \quad \text{if} \quad -P_\mu^2\Psi = M^2\Psi\,.$$

In addition we suppose that the states Φ and Ψ have spin zero.

This situation may occur for example in the case of A^4-coupling if there is an elementary particle of mass m and a stable two particle bound state of the mass M.

In order to describe the bound states we introduce the operator (¹²)

$$B(x, \xi) = TA(x+\xi)A(x-\xi) \tag{2}$$

and define (¹³) incoming fields, according to WIGHTMAN (¹⁴), by

$$\begin{cases} A_{\text{in}}(x) \quad = A(x) + \int \Delta_{\text{Ret}}(m, x-x')j(x')\,\mathrm{d}x'\,,\\ B_{\text{in}}(x, \xi) = B(x, \xi) + \int \Delta_{\text{Ret}}(M, x-x')j(x', \xi)\,\mathrm{d}x'\,, \end{cases} \tag{3}$$

(¹²) ξ may be taken as spacelike or timelike 4-vector, but it is supposed that $\xi^2 \neq 0$.

(¹³) The integral expressions in eq. (3) are to be understood in the sense of weak operator convergence which means that

$$(\Phi, A_{\text{in}}(x)\Psi) = (\Phi, A(x)\Psi) + \int \Delta_{\text{Ret}}(m, x-x')(\Phi, j(x')\Psi)\,\mathrm{d}x'\,,$$

between any normalizable state vectors Φ and Ψ.

(¹⁴) The following definition of the asymptotic fields $A_{\text{in}}(x)$, $A_{\text{out}}(x)$ was first suggested by WIGHTMAN: private communication.

with the current operators

$$(3') \qquad \begin{cases} j(x) = (\Box_x - m^2)A(x)\,, \\ j(x, \xi) = (\Box_x - M^2)B(x, \xi)\,, \end{cases} \qquad \Box_x = \sum \frac{\partial^2}{\partial x_\mu^2}\,.$$

The outgoing fields $A_{\text{out}}(x)$, $B_{\text{out}}(x, \xi)$ are correspondingly defined with the help of the advanced functions $\Delta_{\text{Adv}}(m, x)$ and $\Delta_{\text{Adv}}(M, x)$. The general conditions under which the so defined operators $A_{\substack{\text{out}\\\text{in}}}(x)$ exist have been given and investigated in details by GREENBERG and can easily be extended to the case of the operators $B_{\substack{\text{out}\\\text{in}}}(x, \xi)$ (15,16).

Finally we demand that the incoming field operators $A_{\text{in}}(x)$, $B_{\text{in}}(x, \xi)$ together form an irreducible operator ring and correspondingly the outgoing fields $A_{\text{out}}(x)$, $B_{\text{out}}(x, \xi)$.

So far we have listed the general requirements which we need in the work which follows. We conclude this section by deriving some simple properties of the operators $A_{\substack{\text{out}\\\text{in}}}(x)$, $B_{\substack{\text{out}\\\text{in}}}(x, \xi)$. As a consequence of definition (3) they are solutions of the Klein-Gordon equations for the masses m and M, respectively:

$$(4) \qquad \begin{cases} (\Box_x - m^2)\,A_{\substack{\text{out}\\\text{in}}}(x) = 0\,, \\ (\Box_x - M^2)B_{\substack{\text{out}\\\text{in}}}(x, \xi) = 0\,. \end{cases}$$

We have the invariance properties

$$(5) \qquad \frac{\partial A_{\substack{\text{out}\\\text{in}}}(x)}{\partial x_\mu} = -\,i[P_\mu, A_{\substack{\text{out}\\\text{in}}}(x)]\,, \qquad \frac{\partial B_{\substack{\text{out}\\\text{in}}}(x, \xi)}{\partial x_\mu} = -\,i[P_\mu, B_{\substack{\text{out}\\\text{in}}}(x, \xi)]\,,$$

$$A_{\substack{\text{out}\\\text{in}}}(Lx) = U(L)A_{\substack{\text{out}\\\text{in}}}(x)U(L)^{-1}, \qquad B_{\substack{\text{out}\\\text{in}}}(Lx, L\xi) = U(L)B_{\substack{\text{out}\\\text{in}}}(x, \xi)U(L)^{-1}\,,$$

for an arbitrary Lorentz transformation L. ($U(L)$ denotes the unitary operator transforming $A(x)$ into $A(Lx)$.) These invariance properties are easy to prove if (3) is written in momentum space.

As a consequence of (4), (5) the vacuum expectation values of the incoming fields vanish:

$$(6) \qquad (\Omega, A_{\substack{\text{out}\\\text{in}}}(x)\,\Omega) = 0\,, \qquad (\Omega, B_{\substack{\text{out}\\\text{in}}}(x, \xi)\,\Omega) = 0\,.$$

For example,

$$M^2(\Omega, B_{\text{in}}(x, \xi)\Omega) = \sum \frac{\partial^2}{\partial x_\mu^2}\,(\Omega, B_{\text{in}}(x, \xi)\Omega) = 0\,,$$

because $P_\mu \Omega = 0$.

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(15) O. W. GREENBERG and A. S. WIGHTMAN: preprint, to be published.

(16) In Sect. 2 we will use somewhat more restrictive conditions than Greenberg, for details see reference (20).

Furthermore $A_{\substack{\text{out}\\ \text{in}}}(x)$ and $B_{\substack{\text{out}\\ \text{in}}}(x, \xi)$ satisfy the asymptotic conditions

$$(7)\qquad \left\{\begin{aligned} &\lim_{t\to\pm\infty} (\Phi, A_f(t)\Psi) = (\Phi, A_{\substack{\text{out}\\ \text{in}}f}\Psi)\,,\\ &\lim_{t\to\pm\infty} (\Phi, B_F(t, \xi)\Psi) = (\Phi, B_{\substack{\text{out}\\ \text{in}}F}(\xi)\Psi)\,,\end{aligned}\right.$$

with

$$(7')\qquad \begin{aligned} B_F(t, \xi) &= -i\int\limits_{x_0=t} \mathrm{d}_3x \left\{B(x, \xi)\frac{\partial}{\partial x_0}F^*(x) - F^*(x)\frac{\partial}{\partial x_0}B(x, \xi)\right\},\\ B_{\substack{\text{out}\\ \text{in}}F}(\xi) &= -i\int \mathrm{d}_3x \left\{B_{\substack{\text{out}\\ \text{in}}}(x, \xi)\frac{\partial}{\partial x_0}F^*(x) - F^*(x)\frac{\partial}{\partial x_0}B_{\substack{\text{out}\\ \text{in}}}(x, \xi)\right\},\end{aligned}$$

(correspondingly the definition of A_f and $A_{\substack{\text{out}\\ \text{in}}f}$) for any normalizable solution $f(x)$, $F(x)$ of the Klein-Gordon equation

$$(\Box - m^2)\, f(x) = 0\,,$$

$$(\Box - M^2)F(x) = 0\,.$$

Eq. (7) can be proved by forming the integral expression (7′) for both sides of (3) and taking the limit $t \to \pm\infty$.

2. – Commutation relations for the asymptotic fields.

2·1. *Elementary particles.* – Our first aim is to derive commutation relations for the incoming (or outgoing) fields defined in Sect. 1. We begin with the case of elementary particles and prove the relation ([17])

$$(8)\qquad [A_{\text{in}}(x),\, A_{\text{in}}(y)] = [A_{\text{out}}(x),\, A_{\text{out}}(y)] = i\,\Delta(m, x-y)$$

if $A(x)$ is normalized in the following manner. Let Φ_k be a one-particle state with energy momentum eigenvalue k_μ

$$\left\{\begin{aligned} &P_\mu\Phi_k = k_\mu\Phi_k\,,\\ &(\Phi_k, \Phi_{k'}) = 2k_0^m\,\delta_3(\boldsymbol{k}-\boldsymbol{k}')\,,\end{aligned}\right. \qquad -k_\mu^2 = m^2, \quad k_0^m = |\sqrt{\boldsymbol{k}^2 + m^2}\,|\,.$$

([17]) This was already shown by GREENBERG, reference ([15]) under the assumption that the equal time commutator $[A(x)\dot{A}(y)]_{x_0=y_0}$ is a c-number. Here we use a different method which can be extended to the case of bound states.

Then it follows from translation invariance that

$$(\Omega, A(x)\Phi_k) = c\,\frac{\exp[ikx]}{(2\pi)^{\frac{3}{2}}}\,,$$

with the constant

$$c = (2\pi)^{\frac{3}{2}}(\Omega, A(0)\,\Phi_k)\,.$$

Now $A(x)$ shall be normalized by the condition $c = 1$.

It is easy to determine the matrix elements of $A_{\text{in}}(x)$ between the vacuum state and an arbitrary state vector. From the definition (3) it follows that

$$(\Omega, A_{\text{in}}(x)\Phi_k) = (\Omega, A(x)\Phi_k) = \frac{\exp[ikx]}{(2\pi)^{\frac{3}{2}}}\,,$$

for the one-particle state Φ_k. On the other hand if Φ is an eigenstate of $-P_\mu^2$ with a rest mass $\varkappa^2 \neq m^2$ we have

$$(\Omega, A_{\text{in}}(x)\,\Phi) = 0$$

because

$$(m^2 - \varkappa^2)(\Omega, A_{\text{in}}(x)\,\Phi) = 0$$

follows from (4) and (5).

With this result we can calculate the vacuum expectation value of the commutator (8). We see that

$$(\Omega, A_{\text{in}}(x)A_{\text{in}}(y)\Omega) = \int \frac{d_3k}{2\,k_0^m}\,(\Omega, A_{\text{in}}(x)\Phi_k)(\Phi_k, A_{\text{in}}(y)\Omega) = i\,\Delta^+(m, x-y)\,.$$

Hence,

$$(\Omega, [A_{\substack{\text{out}\\ \text{in}}}(x)\,A_{\substack{\text{out}\\ \text{in}}}(y)]\,\Omega) = i\,\Delta(m, x-y)\,. \tag{9}$$

Now we turn to the operator form (8) and using the conditions of Sect. 1 we shall show that

(i) the commutators of the incoming and the outgoing fields coincide:

$$[A_{\text{out}}(x), A_{\text{out}}(y)] = [A_{\text{in}}(x), A_{\text{in}}(y)] \tag{10}$$

and that

(ii) the commutators of the incoming field is a c-number:

$$[A_{\text{in}}(x), A_{\text{in}}(y)] = (\Omega, [A_{\text{in}}(x), A_{\text{in}}(y)]\,\Omega)\,. \tag{11}$$

Relation (8) is, of course, a consequence of the statements (i), (ii) and Eq. (9).

In order to prove (i) and (ii) it is convenient to expand the field operators with respect to a complete orthonormal system $\{f_\alpha(x)\}$ of the positive frequency solutions of $(\Box - m^2)\, f(x) = 0$:

$$\begin{aligned} A(x) \quad &= \sum f_\alpha(x) A^+_\alpha(x_0) + \sum f^*_\alpha(x) A^-_\mu(x_0)\,, \\ A_{\substack{\text{out}\\ \text{in}}}(x) &= \sum f_\alpha(x) A^+_{\substack{\text{out}\\ \text{in}}\,\alpha} + \sum f^*_\alpha(x) A^-_{\substack{\text{out}\\ \text{in}}\,\alpha}\,. \end{aligned}$$

For the proof of statement (i) we start with the identity (¹⁸)

$$(12) \quad \int \mathrm{d}x \int \mathrm{d}y f^*_\alpha(x)\, f_\beta(y) K^m_x K^m_y\, TA(x)A(y) = \int \mathrm{d}y \int \mathrm{d}x\, f^*_\alpha(x) f_\beta(y) K^m_x K^m_y\, TA(x)A(y)\,,$$

$$K^m_x = \sum \frac{\partial^2}{\partial x^2_\mu} - m^2\,.$$

This relation is not self evident because there are simple examples of pathological fields which do not satisfy eq. (12). But using the fact that as a consequence of causality and spectrum conditions the vacuum expectation values

$$\big(\Omega,\, A(x_1)\cdot\cdot A(x_n)\, T\big(A(x),\, A(y)\big)\, A(y_1)\cdot\cdot A(y_m)\,\Omega\big)$$

are boundary values of Wightman's analytical functions (¹⁹) a more detailed investigation (²⁰) justifies the interchanging of the x- and y-integration in

$$\int \mathrm{d}x \int \mathrm{d}y\, f^*_\alpha(x) f_\beta(y) K^m_x K^m_y (\Omega,\, A(x_1) \ldots A(x_n) T\big(A(x),\, A(y)\big) A(y_1) \ldots A(y_m)\Omega)\,.$$

Then relation (12) holds for every matrix element of $TA(x)A(y)$ because, according to the irreducibility of the operators $A(x)$, any state vector can be written as a linear superposition of vectors $A(x_n) \ldots A(x_1)\Omega$.

Rearranging the integral on the left hand side with the help of Green's theorem we get

$$\begin{aligned} -i \int \mathrm{d}y\, f_\beta(y) K^m_y\, TA(x)A(y) = -i \int \mathrm{d}y_0 \frac{\partial}{\partial y_0} \int \mathrm{d}_3 y\, TA(x)A(y) \overset{\leftrightarrow}{\frac{\partial}{\partial y_0}} f_\beta(y) &= \\ = A(x) A^-_{\text{in}\beta} - A^-_{\text{out}\beta} A(x)&\,, \end{aligned}$$

$$\left(f(x) \overset{\leftrightarrow}{\frac{\partial}{\partial x_0}} g(x) = f(x)\, \frac{\partial g(x)}{\partial x_0} - g(x)\, \frac{\partial f(x)}{\partial x_0} \right).$$

(¹⁸) In the sense of weak operator convergence (compare footnote (¹³)), correspondingly for all following operator expressions containing time integration.

(¹⁹) A. S. WIGHTMAN: *Phys. Rev.*, **101**, 860 (1956).

(²⁰) W. ZIMMERMANN: *Order of integrations in reduction formulae*, unpublished manuscript.

Carrying out the integration over x in the same way, we obtain

$$\int \mathrm{d}x \int \mathrm{d}y\, f^*_\alpha(x) f_\beta(y) K^m_x K^m_y\, TA(x)A(y) = A^-_{\text{out}\beta} A^+_{\text{out}\alpha} - A^-_{\text{out}\beta} A^+_{\text{in}\alpha} - A^+_{\text{out}\alpha} A^-_{\text{in}\beta} + A^+_{\text{in}\alpha} A^-_{\text{in}\beta}\,.$$

For the integral on the right hand side of (12) we have, correspondingly

$$\int \mathrm{d}y \int \mathrm{d}x\, f^*_\alpha(x) f_\beta(y) K^m_x K^m_y\, TA(x)A(y) = A^+_{\text{out}\alpha} A^-_{\text{out}\beta} - A^+_{\text{out}\alpha} A^-_{\text{in}\beta} - A^-_{\text{out}\beta} A^+_{\text{in}\alpha} + A^-_{\text{in}\beta} A^+_{\text{in}\alpha}\,.$$

Inserting these expressions into (12), we get

$$[A^+_{\text{out}\alpha}\, A^-_{\text{out}\beta}] - [A^+_{\text{in}\alpha}\, A^-_{\text{in}\beta}] = 0\,.$$

With the corresponding relations between $A^+_{\substack{\text{out}\\ \text{in}}\alpha}$ and $A^+_{\substack{\text{out}\\ \text{in}}\beta}$ or $A^-_{\substack{\text{out}\\ \text{in}}\alpha}$ and $A^-_{\substack{\text{out}\\ \text{in}}\beta}$ statement (i) follows:

$$[A_{\text{in}}(x), A_{\text{in}}(y)] = [A_{\text{out}}(x), A_{\text{out}}(y)]\,.$$

The second statement (ii) may be derived from the identity

$$(13)\quad \int \mathrm{d}x \int \mathrm{d}y\, f^*_\alpha(x) f_\beta(y) K^m_x K^m_y\, TA(x)\, A(y)\, A(z) = \\ = \int \mathrm{d}y \int \mathrm{d}x\, f^*_\alpha(x) f_\beta(y) K^m_x K^m_y\, TA(x)\, A(y)\, A(z)\,,$$

which can again be rearranged with the help of Green's theorem. The final result is

$$A(z)[A^+_{\text{in}\alpha}\, A^-_{\text{in}\beta}] = [A^+_{\text{out}\alpha}\, A^-_{\text{out}\beta}] A(z)\,.$$

Using statement (i), we get

$$[[A^+_{\text{in}\alpha}\, A^-_{\text{in}\beta}]\, A(z)] = 0\,,$$

and this shows that $[A^+_{\text{in}\alpha}, A^-_{\text{in}\beta}]$ is a c-number because we have assumed that the $A(x)$ form an irreducible operator ring. With the corresponding results for $[A^+_{\text{in}\alpha}, A^+_{\text{in}\beta}]$, $[A^-_{\text{in}\alpha}, A^-_{\text{in}\beta}]$, we have proved the statement (ii).

2·2. *Bound states.* – In the next step we want to derive commutation relations for the field operators $B_{\substack{\text{out}\\ \text{in}}}(x, \xi)$. We begin with the proof of the following statements:

(i) The commutators of the incoming and outgoing fields coincide:

$$[B_{\text{out}}(x, \xi) B_{\text{out}}(y, \eta)] = [B_{\text{in}}(x, \xi) B_{\text{in}}(y, \eta]\,. \tag{14}$$

(ii) The commutator of the incoming field is a c-number:

$$[B_{\text{in}}(x, \xi) B_{\text{in}}(y, \eta)] = (\Omega, [B_{\text{in}}(x, \xi) B_{\text{in}}(x, \eta)]\Omega)\,. \tag{15}$$

We expand the operators B and $B_{\substack{\text{out}\\ \text{in}}}$ with respect to a complete orthonormal system $\{F_\alpha(x)\}$ of the positive frequency solutions of $(\Box - M^2)F(x) = 0$:

$$\begin{aligned} B(x, \xi) &= \sum F_\alpha(x) B_\alpha^+(x_0, \xi) + \sum F_\alpha^*(x) B_\alpha^-(x_0, \xi)\,, \\ B_{\substack{\text{out}\\ \text{in}}}(x, \xi) &= \sum F_\alpha(x) B^+_{\substack{\text{out}\\ \text{in}}\,\alpha}(\xi) + \sum F_\alpha^*(x) B^-_{\substack{\text{out}\\ \text{in}}\,\alpha}(\xi)\,. \end{aligned}$$

The coefficients are determined by (²¹)

$$B_\alpha^\pm(x_0, \xi) = \mp\, i\int \mathrm{d}_3x \left\{B(x, \xi)\frac{\partial}{\partial x_0} F_\alpha^\mp(x) - F_\alpha^\mp(x)\frac{\partial}{\partial x_0} B(x, \xi)\right\},$$

$$F_\alpha^+(x) = F_\alpha(x),\ F_x^-(x) = F_\alpha^*(x),$$

(and similarly for $B^\pm_{\substack{\text{out}\\ \text{in}}\,x}(\xi)$).

As a consequence of causality and spectrum conditions we have the identity

$$\begin{aligned} \int \mathrm{d}x \int \mathrm{d}y\, F_\alpha^*(x) F_\beta(y) K_x^M K_y^M T A(x + \xi) A(x - \xi) A(y + \eta) A(y - \eta) &= \\ = \int \mathrm{d}y \int \mathrm{d}x\, F_\alpha^*(x) F_\beta(y) K_x^M K_y^M T A(x + \xi) A(x - \xi) A(y + \eta) A(y - \eta)\,. \end{aligned} \tag{16}$$

After integration over x_0 and y_0, this relation yields

$$[B^+_{\text{out}\,x}(\xi), B^-_{\text{out}\beta}(\eta)] - [B^+_{\text{in}x}(\xi), B^-_{\text{in}\beta}(\eta)] = 0\,,$$

and similarly for $[B^\pm_{\text{out}\alpha}(\xi), B^\pm_{\text{out}\beta}(\eta)]$. So we have proved that

$$[B_{\text{out}}(x, \xi), B_{\text{out}}(y, \eta)] = [B_{\text{in}}(x, \xi), B_{\text{in}}(y, \eta)]\,.$$

(²¹) If ξ is spacelike we have

$$B_\alpha^-(x_0, \xi) = B_\alpha^+(x_0, \xi)^*\,.$$

This relation is no longer true if ξ is a timelike vector.

Statement (ii) may be derived from

$$(17)\quad \int \mathrm{d}x \int \mathrm{d}y\, F^*_\alpha(x) F_\beta(y) K^M_x K^M_y\, TA(x+\xi)A(x-\xi)A(y+\eta)A(y-\eta)A(z) =$$

$$= \int \mathrm{d}y \int \mathrm{d}x\, F^*_\alpha(x) F_\beta(y) K^M_x K^M_y\, TA(x+\xi)A(x-\xi)A(y+\eta)A(y-\eta)A(z)\,,$$

which yields

$$[B^+_{\mathrm{out}\,\alpha}(\xi)\, B^-_{\mathrm{out}\,\beta}(\eta)]\, A(z) = A(z)[B^+_{\mathrm{in}\,\alpha}(\xi)\, B^-_{\mathrm{in}\,\beta}(\eta)]$$

or

$$[[B^+_{\mathrm{in}\,\alpha}(\xi)\, B^-_{\mathrm{in}\,\beta}(\eta)]\, A(z)] = 0\,.$$

Therefore the commutator $[B_{\mathrm{in}}(x, \xi)\, B_{\mathrm{in}}(y, \eta)]$ is a c-number (statement ii).

Now we can determine the commutator (15) by calculating the vacuum expectation value. For this purpose we determine first the matrix elements of $B_{\mathrm{in}}(x, \xi)$ between the vacuum state and an arbitrary state vector.

If Φ_k is an eigenstate of P_μ belonging to the eigenvalue k_μ and the rest mass $-k^2_\mu = M^2$ we have

$$(\Omega,\, TA(x+\xi)A(x-\xi)\Phi_k) = \frac{\exp[ikx]}{(2\pi)^{\frac{3}{2}}}\, F_k(\xi)\,,$$

with

$$(18)\qquad F_k(\xi) = (2\pi)^{\frac{3}{2}}(\Omega,\, TA(\xi)A(-\xi)\Phi_k)$$

and

$$(\Box_x + M^2)(\Omega,\, TA(x+\xi)A(x-\xi)\Phi_k) = 0\,.$$

From this and the definition (3) of $B_{\mathrm{in}}(x, \xi)$ it follows that

$$(19)\qquad (\Omega,\, B_{\mathrm{in}}(x, \xi)\Phi_k) = (\Omega,\, B(x, \xi)\Phi_k) = \frac{\exp[ikx]}{(2\pi)^{\frac{3}{2}}}\, F_k(\xi)\,.$$

On the other hand if Φ is an eigenstate belonging to the eigenvalue k_μ with the rest mass $-k^2_\mu \neq M^2$ we get from (5):

$$(k^2_\mu + M^2)(\Omega,\, B_{\mathrm{in}}(x, \xi)\Phi) = -(\Box_x - M^2)(\Omega,\, B_{\mathrm{in}}(x, \xi)\Phi) = 0$$

hence

$$(\Omega,\, B_{\mathrm{in}}(x, \xi)\Phi) = 0\,.$$

Therefore we have

$$(\Omega,\, B_{\mathrm{in}}(x, \xi)B_{\mathrm{in}}(y, \eta)\Omega) = \int \frac{\mathrm{d}_3k}{2k^M_0}(\Omega,\, B_{\mathrm{in}}(x, \xi)\Phi_k)(\Phi_k,\, B_{\mathrm{in}}(y, \eta)\Omega) =$$

$$= \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}_3k}{2k^M_0}\, F_k(\xi)F_k(\eta)\exp[ik(x-y)]\,,$$

where we have used the invariance under space time reflection for

$$(\Phi_k,\, TA(\eta)A(-\eta)\Omega) = (\Omega,\, TA(\eta)A(-\eta)\Phi_k) = F_k(\eta)\,. \tag{20}$$

Introducing the Fourier transform $B_{\text{in}}(k, \xi)$, we have

$$B_{\text{in}}(x, \xi) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}_4 k \exp[ikx]\delta(k^2 + M^2)B_{\text{in}}(k, \xi)\,,$$

and defining creation and annihilation operators

$$\begin{aligned} B^+_{\text{in}}(\boldsymbol{k}, \xi) &= B_{\text{in}}(k, \xi)\,, && \text{for } k_0 = +|\sqrt{\boldsymbol{k}^2 + M^2}|\,,\\ B^-_{\text{in}}(\boldsymbol{k}, \xi) &= B_{\text{in}}(-k, \xi)\,, && \text{for } k_0 = -|\sqrt{\boldsymbol{k}^2 + M^2}|\,, \end{aligned}$$

we get from (19) and (15) the final commutation relations

$$[B^+_{\text{in}}(\boldsymbol{k}, \xi)B^-_{\text{in}}(\boldsymbol{p}, \eta)] = F_k(\xi)F_p(\eta)2k_0^M\delta_3(\boldsymbol{k} - \boldsymbol{p})\,, \tag{21}$$

$$[B^\pm_{\text{in}}(\boldsymbol{k}, \xi)B^\pm_{\text{in}}(\boldsymbol{p}, \eta)] = 0\,. \tag{22}$$

Finally we only mention that the same methods used in this Section to prove the commutation relations (8) and (21) may be applied in order to derive

$$[A_{\text{in}}(x),\, B_{\text{in}}(y, \eta)] = 0\,. \tag{23}$$

3. – Derivation of an S-matrix formula.

The operators $B_{\substack{\text{out}\\ \text{in}}}(x, \xi)$ depend on the relative co-ordinate ξ of the bound state. For the S-matrix we are only interested in the center of mass motion of the bound states and want to carry out the limit $\xi \to 0$. Therefore we define operators $B_{\substack{\text{out}\\ \text{in}}}(x)$ by

$$\left\{\begin{aligned} B_{\substack{\text{out}\\ \text{in}}}(x) &= \lim_{\xi\to 0}\frac{B_{\substack{\text{out}\\ \text{in}}}(x, \xi)}{F_0(\xi)}\,,\\ F_0(\xi) &= (2\pi)^{\frac{3}{2}}(\Omega,\, TA(\xi)A(-\xi)\Phi_0)\,, \end{aligned}\right. \tag{24}$$

where Φ_0 is the bound state at rest.

In order to prove the existence of the limits $\xi \to 0$ in (24) we divide (21) by $F_k(\xi)$ and get

$$\left[\frac{B^+_{\text{in}}(k, \xi)}{F_k(\xi)},\, B^-_{\text{in}}(p, \eta)\right] = F_k(\eta)\delta_3(\boldsymbol{k} - \boldsymbol{p})\,.$$

Differentiation with respect to ξ_μ yields

$$\left[\frac{\partial}{\partial \xi_\mu}\,\frac{B^+_{\text{in}}(k,\xi)}{F_k(\xi)},\, B^-_{\text{in}}(p,\eta)\right] = 0\,,$$

and correspondingly from (22), (23)

$$\left[\frac{\partial}{\partial \xi_\mu}\,\frac{B^+_{\text{in}}(k,\xi)}{F_k(\xi)},\, B^+_{\text{in}}(p,\eta)\right] = 0\,,$$

$$\left[\frac{\partial}{\partial \xi_\mu}\,\frac{B^+_{\text{in}}(k,\xi)}{F_k(\xi)},\, A^\pm_{\text{in}}(p)\right] = 0\,.$$

Thus $(\partial/\partial\xi_\mu)(B^+_{\text{in}}(k,\xi)/(F_k(\xi))$ commutes with all operators $A_{\text{in}}(x)$ and $B_{\text{in}}(y,\eta)$. Since we have assumed that the operators $A_{\text{in}}(x)$, $B_{\text{in}}(y,\eta)$ together form an irreducible operator ring $(\partial/\partial\xi_\mu)(B^+_{\text{in}}(k,\xi)/F_k(\xi))$ is a c-number:

$$\frac{\partial}{\partial \xi_\mu}\,\frac{B^+_{\text{in}}(k,\xi)}{F_k(\xi)} = \frac{\partial}{\partial \xi_\mu}\,\frac{(\Omega, B^+_{\text{in}}(k,\xi)\Omega)}{F_k(\xi)} = 0\,,$$

because of (6). Therefore the expression

$$\frac{B^+_{\text{in}}(k,\xi)}{F_k(\xi)} = B^\pm_{\text{in}}(k)\,,$$

is independent of ξ and

(25) $$B_{\text{in}}(k) = \lim_{\xi\to 0} \frac{B_{\text{in}}(k,\xi)}{F_0(\xi)}\,,$$

does exist because

$$\frac{F_k(\xi)}{F_0(\xi)} = \frac{F(k\xi, \xi^2)}{F(0, \xi^2)}\,,$$

is in the limit $\xi \to 0$ independent of k:

$$\lim_{\xi\to 0} \frac{F_k(\xi)}{F_0(\xi)} = 1\,.$$

From (21), (22), (23) we get:

$$[B^+_{\text{in}}(\boldsymbol{k}), B^-_{\text{in}}(\boldsymbol{p})] = 2k_0^M\,\delta_3(\boldsymbol{k}-\boldsymbol{p})\,,$$

$$[B^\pm_{\text{in}}(\boldsymbol{k}), B^\pm_{\text{in}}(\boldsymbol{p})] = 0\,,$$

$$[B^\pm_{\text{in}}(\boldsymbol{k}), A_{\text{in}}(y)] = 0\,.$$

Hence

$$[B_{\substack{\text{out}\\\text{in}}}(x),\, B_{\substack{\text{out}\\\text{in}}}(y)] = i\Delta(M,\, x-y)\,, \tag{26}$$

$$[B_{\substack{\text{out}\\\text{in}}}(x),\, A_{\substack{\text{out}\\\text{in}}}(y)] = 0\,. \tag{27}$$

Further we have

$$(\Box - M^2)\, B_{\substack{\text{out}\\\text{in}}}(x) = 0$$

and

$$(\Omega,\, B_{\substack{\text{out}\\\text{in}}}(x)\Phi_k) = \frac{\exp\,[ikx]}{(2\pi)^{\frac{3}{2}}}\,, \qquad k^2 + M^2 = 0\,,$$

if Φ_k is the bound state with four momentum k_μ. $B_{\substack{\text{out}\\\text{in}}}(x)$ is given in terms of $A(x)$ by

$$B_{\substack{\text{out}\\\text{in}}}(x) = \lim_{\xi\to 0}\frac{1}{F_0(\xi)}\left\{TA(x+\xi)A(x-\xi) + \int \mathrm{d}x'\,\Delta_{\substack{\text{Adv}\\\text{Rct}}}(M,\, x-x')J(x',\, \xi)\right\}, \tag{28}$$

$$J(x,\, \xi) = (\Box_x - M^2)TA(x+\xi)A(x-\xi)\,.$$

According to (25) $B^{\pm}_{\text{in}}(k, \xi)$ is a multiple of $B_{\text{in}}(k)$. From this fact we conclude that the operators $A_{\text{in}}(x)$, $B_{\text{in}}(y)$ together already form an irreducible operator ring. Therefore the whole Hilbert space $\mathcal{H}$ can be built up by the creation operators

$$A^{-}_{\substack{\text{out}\\\text{in}}}(k)\,, \qquad B^{-}_{\substack{\text{out}\\\text{in}}}(k)$$

of incoming or outgoing elementary particles or bound states of momentum k. The state vectors

$$\Phi^{k_1 \dots k_n}_{\text{in}} = A^{-}_{\text{out}}(\boldsymbol{k}_1) \dots A^{-}_{\text{in}}(\boldsymbol{k}_l) B^{-}_{\text{in}}(\boldsymbol{k}_{l+1}) \dots B^{-}_{\text{in}}(\boldsymbol{k}_n)\,\Omega \tag{29}$$

as well as

$$\begin{gathered}\Phi^{k_1 \dots k_n}_{\text{out}} = A^{-}_{\text{out}}(\boldsymbol{k}_1) \dots A^{-}_{\text{out}}(\boldsymbol{k}_l) B^{-}_{\text{out}}(\boldsymbol{k}_{l+1}) \dots B^{-}_{\text{out}}(\boldsymbol{k}_n)\,\Omega\,, \\ (k_i^2 = -m^2 \quad \text{for} \quad i \leqslant l\,, \quad k_i^2 = -M^2 \quad \text{for} \quad i > l)\end{gathered} \tag{30}$$

form a complete orthonormal system of $\mathcal{H}$. The S-matrix is defined as the operator transforming the incoming into the outgoing states:

$$(\Phi^{p_1 \dots p_k}_{\text{in}},\, S\Phi^{q_1 \dots q_l}_{\text{in}}) = (\Phi^{p_1 \dots p_k}_{\text{out}},\, \Phi^{q_1 \dots q_l}_{\text{in}})\,. \tag{31}$$

Since both systems (29) and (30) form a complete basis of the Hilbert space the S-matrix is unitary.

It is possible to express the S-matrix by the vacuum expectation values of T-products only. To show this we derive from (7) the following reduction formula (22)

$$[ST(x_1 \dots x_k), A^{\pm}_{\text{in}\alpha}] = \pm i\int dz f^{\mp}_{\alpha}(z) K^m_z ST(x_1 \dots x_k z)\,, \tag{32}$$

$$[ST(x_1 \dots x_k), B^{\mp}_{\text{in}\alpha}(\zeta)] = \pm i\int dz F^{\mp}_{\alpha}(z) K^M_z ST(x_1 \dots x_k, z-\zeta, z+\zeta)\,, \tag{33}$$

with

$$T(x_1 \dots x_n) = TA(x_1) \dots A(x_n), \qquad \begin{array}{l} f^+_\alpha(z) = f_\alpha(z), \\ f^-_\alpha(z) = f^*_\alpha(z). \end{array}$$

If we now insert plane waves instead of the wave packets $f_\alpha(x)$ we get

$$[ST(x_1 \dots x_k), A^*_{\text{in}}(k)] = -\frac{i\varepsilon(k)}{(2\pi)^{\frac{3}{2}}}\int dz \exp[ikz] K^m_z ST(x_1 \dots x_k z)\,, \tag{34}$$

$$\begin{aligned}[ST(x_1 \dots x_k), B^*_{\text{in}}(k)] &= \\ &= -\frac{i\varepsilon(k)}{(2\pi)^{\frac{3}{2}} F_k(\zeta)}\int dz \exp[ikz] K^M_z ST(x_1 \dots x_k, z-\zeta, z+\zeta)\,.\end{aligned} \tag{35}$$

Since the state vectors (29) form a complete basis S can be expanded with respect to the incoming fields:

$$\begin{aligned} S = \sum_{n,m=0}^{\infty} \frac{(-i)^{m+n}}{m!\,n!}\int & dk_1 \dots dk_m\, dl_1 \dots dl_n\, c(k_1 \dots k_m; l_1 \dots l_n) \cdot \\ & \cdot \prod_{i=1}^{m} \delta(k_i^2 + m^2) \prod_{j=1}^{n} \delta(k_j^2 + M^2) :A_{\text{in}}(k_1) \dots A_{\text{in}}(k_m) B_{\text{in}}(l_1) \dots B_{\text{in}}(l_n):\,, \end{aligned} \tag{36}$$

where the coefficients are

$$\left\{ \begin{aligned} & c(k_1 \dots k_m; l_1 \dots l_n) = \\ & \quad = \varepsilon(k_1) \dots \varepsilon(k_m) \varepsilon(l_1) \dots \varepsilon(l_n) (\Omega, [\dots[S, A^*_{\text{in}}(k_1)] \dots A^*_{\text{in}}(k_m)] B^*_{\text{in}}(l_1)] \dots B^*_{\text{in}}(l_n)]\Omega) = \\ & \quad = \frac{1}{(2\pi)^{\frac{3}{2}(m+n)} F_{l_1}(\zeta_1) \dots F_{l_n}(\zeta_n)} \int dy_1 \dots dy_m\, dz_1 \dots dz_n \exp\left[i\left(\sum k_i y_i + \sum l_j z_j\right)\right] \cdot \\ & \cdot K^m_{y_1} \dots K^m_{y_m} K^M_{z_1} \dots K^M_{z_n} (\Omega, T(y_1 \dots y_m, z_1+\zeta_1, \dots z_n+\zeta_n, z_1-\zeta_1, \dots, z_n-\zeta_n)\Omega)\,. \end{aligned} \right.$$

---

(22) All reduction formulae in H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, 1, 425 (1955), Sect. 2 are contained in the single formula (32) which was given by K. SYMANZIK: unpublished.

The last line of (37) follows by iterating (34), (35) and taking the vacuum expectation value. Transforming (37) into co-ordinate space we get (23)

$$
(38)\quad \left\{\begin{aligned}
&S = \sum_{m,n=0}^{\infty} \frac{(-i)^{m+n}}{m!\,n!} \lim_{\zeta\to 0} \frac{1}{F_0(\zeta)^n} \int \mathrm{d}y_1 \dots \mathrm{d}y_m \,\mathrm{d}z_1 \dots \mathrm{d}z_n K^m_{y_1} \dots K^m_{y_m} K^M_{z_1} \dots K^M_{z_n} \cdot \\
&\cdot (\Omega,\, T(y_1 \dots y_m,\, z_1+\zeta, \dots, z_n+\zeta,\, z_n-\zeta, \dots z_n-\zeta)\Omega) : \\
&\qquad\qquad : A_{\text{in}}(y_1) \dots A_{\text{in}}(y_m) B_{\text{in}}(z_1) \dots B_{\text{in}}(z_n) : .
\end{aligned}\right.
$$

The function $F_k(\zeta)$ which enters in the expansions (37) and (38) can easily be expressed by vacuum τ-functions (*i.e.*, vacuum expectation values of T-products) if we take the vacuum expectation value of reduction formula (35) for $k=2$, $x_1=\xi$, $x_2=-\xi$:

$$
(39)\quad F_k(\xi) F_k(\eta) = -i \int \mathrm{d}y \exp[iky] K^M_y (\Omega,\, TA(\xi)A(-\xi)A(y+\eta)A(y-\eta)\Omega) .
$$

Putting $\xi=\eta$ we get

$$
(40)\quad F_k(\xi)^2 = -i \int \mathrm{d}x \exp[ikx] K^M_x (\Omega,\, TA(\xi)A(-\xi)A(x+\xi)A(x-\xi)\Omega) .
$$

Inserting (40) in (37) or (38) the S-matrix is completely given by the vacuum τ-functions.

4. - Local field operators for bound states.

In this section we want to take the limit $\xi \to 0$ of the operator

$$
B(x;\xi) = TA(x+\xi)A(x-\xi)
$$

itself. We assume the existence of

$$
(41)\quad B(x) = \lim_{\xi\to 0} \frac{TA(x+\xi)A(x-\xi) - (\Omega,\, TA(\xi)A(-\xi)\Omega)}{(2\pi)^{\frac{3}{2}}(\Omega,\, TA(\xi)A(-\xi)\Phi_0)} ,
$$

---

(23) One gets corresponding expansions of the T-product replacing S by $ST(x_1 \dots x_k)$ and

$$
T(y_1 \dots y_m,\, z_1+\zeta_1, \dots, z_n+\zeta_n,\, z_1-\zeta_1, \dots, z_n-\zeta_n) ,
$$

by

$$
T(x_1 \dots x_k,\, y_1 \dots y_m,\, z_1+\zeta_1, \dots, z_n+\zeta_n,\, z_1-\zeta_1, \dots, z_n-\zeta_n) ,
$$

in the expansions (37) and (38).

which in a formal sense can be written as (²⁴)

$$B(x) = a^{-\frac{1}{2}} A(x)^2 - a^{-\frac{1}{2}} b \tag{42}$$

with the two (probably divergent) renormalization constants a and b

$$\left\{\begin{aligned} a &= -i\int \exp\,[-iMx_0](\Omega,\, TA(0)^2A(x)^2\Omega)\,\mathrm{d}x_0\,, \\ b &= (\Omega,\, A(0)^2\Omega)\,. \end{aligned}\right. \qquad \text{(according to (40))} \tag{43}$$

The operator $B(x)$ is, of course, invariant under the inhomogeneous Lorentz group and commutes with $B(y)$ and $A(y)$ if $(x-y)^2 > 0$. Furthermore we get from (7) or (28) the asymptotic conditions

$$\lim_{t\to\pm\infty} (\Phi,\, B_F(t)\Psi) = (\Phi,\, B_{\substack{\text{out}\\ \text{in}}F}\Psi) \tag{44}$$

with

$$B_F(t) = -i\int_{x_0=t} \mathrm{d}_3x\, B(x)\, \frac{\overleftrightarrow{\partial}}{\partial x_0}\, F^*(x)\,,$$

for any normalizable solution of $(\Box_x - M^2)F(x) = 0$.

Therefore the field operators $A(x)$ and $B(x)$ together satisfy the three principles of invariance, causality and asymptotic conditions exactly in the same formulation which was given in an earlier paper for elementary particles only (²⁵). From these principles we get in the usual way an expansion of the S-matrix:

$$\left\{\begin{aligned} S = \sum_{m,n=0}^{\infty} \frac{(-i)^{m+n}}{m!\,n!}\int &\mathrm{d}y_1 \ldots \mathrm{d}y_m\, \mathrm{d}z_1 \ldots \mathrm{d}z_n\, K_{y_1}^m \ldots K_{y_m}^m K_{z_1}^M \ldots K_{z_n}^M \cdot \\ &\cdot (\Omega,\, TA(y_1) \ldots A(y_m)B(z_1) \ldots B(z_n)\Omega) \colon A_{\text{in}}(y_1) \ldots A_{\text{in}}(y_m)B_{\text{in}}(z_1) \ldots B_{\text{in}}(z_n) \colon, \end{aligned}\right. \tag{45}$$

which may also be obtained directly from (38) taking the limit $\zeta \to 0$.

We remark that there remains at least one formal difference between the case of two independent elementary particles and the case of one elementary and one composite particle. If $B(x)$ belongs to a bound state composed of elementary particles described by a field $A(x)$, it is possible to represent $B(x)$

(²⁴) For the case of A^4-coupling at least in perturbation theory the matrix elements of $B(x)$ can be made finite to any order of the coupling constant if a and b are determined by (43).

(²⁵) H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **1**, 425 (1955). Although there only the case of one scalar field was considered the generalization to several fields is obvious.

—roughly speaking—as a polynomial in $A(x)$ (in our model Eq. (42)). We may impose this relationship as an additional condition beyond the principles of invariance, causality and the asymptotic condition in order to exclude the case of two elementary particles.

5. - Examples.

In this Section we shall give simple examples for the expansions of the S-matrix which we have derived in the last sections. Instead of the model of one scalar field we consider the more interesting case of a charged scalar « nucleon » field $\psi(x)$ interacting with a neutral scalar « meson » field $A(x)$. We assume that there are just three non-vanishing discrete eigenvalues m^2, $\varkappa^2$ and M^2 of $-P_\mu^2$: m^2 belongs to the one nucleon states, $\varkappa^2$ to the one meson states and M^2 to a stable bound state Ψ of charge two and spin zero (« deuteron » state). As a consequence of charge conservation

$$(\Omega,\, \psi(x)\Psi) \equiv 0$$

vanishes identically for a deuteron state Ψ. Under the corresponding assumptions, as formulated in Sect. 1 for the case of one scalar field, we can apply the methods developed in the last sections. As an example for the S-matrix expansions (36), (38) and (45) we consider the special scattering process with one deuteron and one meson of momentum q_μ resp k_μ in the initial state and two outgoing nucleons with momentum p_μ^1 and p_μ^2 in the final state:

$$(36') \quad (\Phi_{\text{in}}^{p_1 p_2},\, S\Phi_{\text{in}}^{q,k}) = \frac{1}{(2\pi)^6 F_q(\zeta)} \int dx_1\, dx_2\, dy\, dz \exp\left[i(qz + ky - p_1x_1 - p_2x_2)\right] \cdot \\ \cdot K_{x_1}^m K_{x_2}^m K_y^\varkappa K_z^M (\Omega,\, T\psi(x_1)\psi(x_2)A(y)\overline{\psi}(z+\zeta)\overline{\psi}(z-\zeta)\Omega)\,,$$

$$(38') \quad = \lim_{\zeta\to 0} \frac{1}{(2\pi)^6 F_0(\zeta)} \int dx_1\, dx_2\, dy\, dz \exp\left[i(qz + ky - p_1x_1 - p_2x_2)\right] \cdot \\ \cdot K_{x_1}^m K_{x_2}^m K_z^\varkappa K_y^M (\Omega,\, T\psi(x_1)\psi(x_2)A(y)\overline{\psi}(z+\zeta)\overline{\psi}(z-\zeta)\Omega)\,,$$

$$(45') \quad = \int dx_1\, dx_2\, dy\, dz \exp\left[i(qz + ky - p_1x_1 - p_2x_2)\right] \cdot \\ \cdot K_{x_1}^m K_{x_2}^m K_y^\varkappa K_z^M (\Omega,\, T\psi(x_1)\psi(x_2)A(y)\overline{B}(z)\Omega)\,,$$

$$p_1^2 = p_2^2 = -m^2\,, \qquad q^2 = -M^2\,, \qquad k^2 = -\varkappa^2\,.$$

The function $F_q(\zeta)$ is given by

$$F_q(\zeta) = (2\pi)^{\frac{3}{2}} (\Omega,\, T\psi(\zeta)\psi(-\zeta)\Phi_q) = (2\pi)^{\frac{3}{2}} (\Phi_q,\, T\psi(\zeta)\psi(-\zeta)\Omega)$$

and satisfies

$$F_q(\zeta)^2 = -i(2\pi)^{\frac{3}{2}}\int \mathrm{d}z \exp[iqz]\, K_z^M(\Omega, T\psi(\zeta)\psi(-\zeta)\overline{\psi}(z+\zeta)\overline{\psi}(z-\zeta)\Omega)\,.$$

The deuteron field operator $B(z)$ is defined by

$$B(z) = \lim_{\zeta\to 0} \frac{T\psi(z+\zeta)\psi(z-\zeta)}{F_0(\zeta)}\,,$$

where $F_0(\zeta)$ belongs to the deuteron state Ψ_0 at rest. In a formal sense $B(z)$ may be written as

$$B(z) = c^{-\frac{1}{2}}\psi(z)^2$$

with the renormalization constant

$$c = (2\pi)^3(\Omega, \psi(0)^2\Psi_0)^2 = -\,i\int \exp[-iMx_0](\Omega, T\psi(0)^2\psi(x)^2\Omega)\,\mathrm{d}x\,.$$

Finally we remark that expressions (36′), (38′) and (45′) hold also if the model contains additional stable bound states. In this case the incoming nucleon, meson and deuteron fields are, of course, not irreducible in the whole Hilbert space, and the S-matrix expansions contain in addition the incoming fields belonging to the higher bound states. But if the state vectors $\Phi_{\text{in}}^{k_1\ldots k_2}$ and $\Phi_{\text{out}}^{p_1\ldots p_2}$ contain only nucleon, meson and deuteron states, expressions for the S-matrix elements $(\Phi_{\text{out}}^{p_1\ldots p_2}, \Phi_{\text{in}}^{k_1\ldots k_2})$ can be derived from the asymptotic properties of $A_{\substack{\text{out}\\ \text{in}}}$, $\Psi_{\substack{\text{out}\\ \text{in}}}$ and $B_{\substack{\text{out}\\ \text{in}}}$ in a similar way as in Sect. **3**.

* * *

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Proof of Dispersion Relations in Quantized Field Theories*

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The problem of deriving dispersion formulas is reduced to that of the analytic continuation of all functions which are regular in certain domains in the space of several complex variables. The dispersion relations for pion-nucleon scattering are proven for momentum transfers in the center-of-mass system which are smaller than $2\sqrt{2}M_\pi$. This limit can be improved by further analytic continuation. By using causality and spectral conditions the dispersion formulas for forward nucleon-nucleon scattering could be derived only under the unphysical condition $M_\pi > (\sqrt{2}-1)M_N$. We cannot exclude the possibility that this restriction is weakened by taking into account all symmetry properties of the complete four-body Green's function. The situation is similar for the representation of the meson-nucleon vertex function taken on the mass shell of the nucleons. In this case an example of R. Jost shows that the validity of the dispersion formula cannot be guaranteed on the basis of causality, spectrum, and symmetry properties.

1. INTRODUCTION

THE purpose of the present article is to derive some analytic properties of scattering amplitudes on the basis of general assumptions underlying local relativistic quantum field theories. Most of these analytic properties are usually expressed in terms of Hilbert relations, which have come to be called dispersion relations. Such equations have been obtained for various physical processes, and their importance lies in the fact that in several cases they can be directly tested by experiments.[1] In other cases it is at least possible to extract approximate relations between observable quantities. Hence the dispersion formulas make it possible to test experimentally some aspects of the basic "axioms" mentioned earlier.

We shall not give here a detailed discussion of these axioms, since it may be found elsewhere.[2] They consist essentially of the following assumptions,

(a) The existence of linear field operators in Hilbert space,

(b) the transformation laws of these fields under the transformations of the inhomogeneous Lorentz group,

(c) the asymptotic condition, and

(d) the so-called causality condition, which implies that the commutators (or anticommutators) of two field operators shall vanish if these fields are taken at points which have a finite space-like separation.

For the derivation of dispersion relations we need a few additional assumptions about the existence of Fourier transforms and some known properties of the spectrum of intermediate states for the system in question. These will be discussed later.

On the basis of the axioms (a), $\cdots$, (d), one can derive representations for the elements of the scattering matrix, and we shall be especially concerned with the amplitude for the elastic scattering of particles with finite rest mass. Let k and k' (p and p') be the four-momenta of the projectile (target) before and after scattering. In a special Lorentz frame[1] where $\mathbf{p}+\mathbf{p}'=0$, we consider the scattering amplitude T as a function of the projectile energy $\omega=k_0=k_0'$ for fixed finite values of $\Delta=\frac{1}{2}|\mathbf{k}'-\mathbf{k}|$ (2Δ is the amount of the momentum transfer; we disregard here possible charge and spin variables). We are interested in the analytic properties of $T(\omega,\Delta^2)$ as a function of ω, but the representation obtained from the axioms is valid only for ***real*** ω with $\omega^2 \geq m^2+\Delta^2$, where m is the rest mass of the projectile. (Throughout this paper we set $\hbar=c=1$.) In order to continue the scattering amplitude into the complex ω plane, we must consider it as a function of ω and other variables simultaneously. Thus we are led to use the tools of the theory of functions of several complex variables.

For the case of pion-nucleon scattering a mathematically rigorous proof of dispersion relations for non-forward scattering has been given by Bogoliubov.[3] His proof is valid for values of Δ^2 which are smaller than $\Delta_{\max}{}^2=(m/m+\mu)\mu^2$, where μ is the pion and m is the nucleon mass. Bogoliubov avoids the explicit use of the theory of functions of several complex variables by employing parametrizations and using distribution methods. This makes the proof very involved, and it seems difficult to generalize it to other processes with qualitatively different properties of the spectrum. Because of these difficulties we think that it may be worthwhile to present a different approach in which the essential part of the proof of dispersion formulas is reduced to the problem of finding the *envelope of holomorphy* of a certain type of domain in the space of several complex variables.[4,5] The *envelope of holomorphy* $E(D)$ of a domain D is the intersection of the *domains of holomorphy* of *all* functions which are regular (holomorphic) in D. The *domain of holomorphy* D_f of a function f, which is analytic in D, is the largest domain into which f may be continued. In the case of one complex variable there exists for every domain D a function $f(z)$ such that $D=D_f$, and hence we have $E(D)=D$. But in the space of two or more complex variables the situation is different and we have the remarkable fact that there exist domains D such that *all* functions which are analytic in D may be continued simultaneously into a larger domain. The largest domain into which *all* these functions can be continued is the envelope of holomorphy of D.

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[1] For references see, for instance, Goldberger, Nambu, and Oehme, Ann. phys. 2, 226 (1957).

[2] R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd., 29, No. 12 (1955); Lehmann, Symanzik, and Zimmermann, Nuovo cimento 1, 205 (1955); Bogoliubov, Medvedev, and Polivanov, lecture notes, translated at the Institute for Advanced Study, Princeton, 1957 (unpublished); A. S. Wightman, preprint of Lille Conference talk, June, 1957 (this paper contains further references).

[3] N. N. Bogoliubov (private communication, February 1957). We would like to thank Professor Bogoliubov for informing us about the details of his proof.

For special domains D, which possess certain symmetry properties, the corresponding $E(D)$ is known.[6] There are also methods for the construction of $E(D)$ for an arbitrary D.[7] The domain D which appears in the proof of dispersion formulas has certain invariance properties and we hope that future mathematical work will make it possible to find the corresponding envelope $E(D)$. In the present paper we consider only a suitable subdomain D_S which is a generalized semitube. For this semitube we can construct the envelope of holomorphy and obtain thus a proof of certain dispersion relations for restricted values of Δ^2. In the case of pion-nucleon scattering we have for instance $\Delta^2<2\mu^2$. The semitube method is not sufficient to prove dispersion relations for nucleon-nucleon scattering; it would only suffice if the mass ratio μ/m where larger than $(\sqrt{2}-1)$. However, this limit is due to such points on the surface of $E(D_S)$ which can be shown to be also surface points of $E(D)$. Hence we cannot hope to continue further on the basis of local commutativity and the support properties derived from the spectral conditions. But there are certain symmetry properties of the four-body Green's function which we have not used, and we cannot exclude the possibility that these permit a continuation beyond $E(D)$ in the relevant region.[8]

The situation is very similar in the case of the meson-nucleon vertex function $F[(k-p)^2,\, k^2,\, p^2]=\langle k|\,j(0)\,|p\rangle$. We can assure the existence of the representation $[k^2<(m+\mu)^2,\, p^2<(m+\mu)^2]$

$$F[z,k^2,p^2]=\frac{z^N}{\pi}\int_{(3\mu)^2}^{\infty} d\sigma^2\frac{\rho(\sigma^2,k^2,p^2)}{\sigma^{2N}(\sigma^2-z)}+\sum_{n=0}^{N-1} C_n(k^2,p^2)z^n$$

only for $k^2+p^2<(m+\mu)^2$. In order to reach the mass shell $k^2=p^2=m^2$, we would have to require $\mu>(\sqrt{2}-1)m$.

See Sec. 4 for more detailed discussions of the restrictions in mass ratios and momentum transfer.

2. CONSTRUCTION OF DISPERSION FORMULAS

As an example we consider the elastic scattering of neutral scalar bosons of equal mass. If $S(k',p',k,p)$ describes the relevant S-matrix element for this process, we introduce the usual causal amplitude M_r by the relation

$$S(k',p',k,p)=\langle k'|k\rangle\langle p'|p\rangle+i(2\pi)^4\delta(k'+p'-k-p)\times(16k_0'k_0p_0'p_0)^{-\frac{1}{2}}M_r(k+k';\,p',\,p),\quad(2.1)$$

where k,p and k',p' denote the momenta of the particles before and after scattering. Then, by standard methods,[9] we obtain for M_r the representations

$$M_r(k+k';\,p',\,p)=2(p_0p_0')^{\frac{1}{2}}i\int d^4x\, e^{\frac{1}{2}i(k+k')\cdot x}\eta(x_0)\times\langle p'|[j(\tfrac{1}{2}x),\,j(-\tfrac{1}{2}x)]|p\rangle+P(k+k';\,p',\,p),\quad(2.2)$$

where $j(x)$ may be defined in terms of the Heisenberg fields $\phi(x)$ by $j(x)=(\Box+m^2)\phi(x)$ and where P is a real polynomial in the components of $k+k'$ with arbitrary coefficients depending on p' and p. In addition to M_r we introduce the corresponding advanced amplitude M_a by

$$M_a(k+k',\,p',\,p)=-2(p_0p_0')^{\frac{1}{2}}i\int d^4x\, e^{\frac{1}{2}i(k+k')\cdot x}\eta(-x_0)\times\langle p'|[j(\tfrac{1}{2}x),\,j(-\tfrac{1}{2}x)]|p\rangle+P(k+k';\,p',\,p).\quad(2.3)$$

Then the dispersive and absorptive parts of M_r may be expressed in the form

$$D=\tfrac{1}{2}(M_r+M_a),\quad A=\frac{1}{2i}(M_r-M_a);\quad(2.4)$$

and are real functions of $k+k'$, p' and p. It is convenient for our further discussion to go into the special system in which

$$\tfrac{1}{2}(k+k')=\{\omega,\,(\omega^2-E_\Delta{}^2)^{\frac{1}{2}}\mathbf{e}\};\quad p'=p_\Delta,\quad p=p_{-\Delta},$$

where

$$p_\Delta=\{E_\Delta,\boldsymbol{\Delta}\};\quad \boldsymbol{\Delta}\cdot\mathbf{e}=0;\quad E_\Delta\equiv(m^2+\Delta^2)^{\frac{1}{2}};\quad |\mathbf{e}|=1.$$

In this system the amplitudes M_r and M_a can be

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[4] H. Behnke and P. Thullen, *Theorie der Funktionen mehrerer komplexer Veränderlichen*, Ergebnisse der Mathematische Wissenschaften (Verlag Julius Springer, Berlin and Chelsea Publishing Company, New York, 1934), Vol. 3, No. 3.

[5] S. Bochner and W. T. Martin, *Several Complex Variables* (Princeton University Press, Princeton, 1948).

[6] Hans J. Bremermann, Math. Ann. 127, 406 (1954).

[7] Hans J. Bremermann, "Construction of the Envelopes of Holomorphy of Arbitrary Domains" (to be published).

[8] We would like to thank Professor R. Jost and Professor H. Lehmann for enlightening discussions concerning this point.

[9] See, for example, Lehmann *et al.* and Bogoliubov *et al.*, reference 2.

written as

$$M_{r,a}(\omega,\Delta^2)=D(\omega,\Delta^2)\pm iA(\omega,\Delta^2)$$
$$=\pm 2E_\Delta i\int d^4x\,\exp[i\omega x_0-i(\omega^2-E_\Delta{}^2)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}]$$
$$\times\eta(\pm x_0)\langle p_\Delta|[j(\tfrac{1}{2}x),\,j(-\tfrac{1}{2}x)]|p_{-\Delta}\rangle$$
$$+\sum_{n=0}^{N-1}C_n(\Delta^2)\omega^{2n}.\quad(2.5)$$

One can easily prove that for fixed Δ^2 the dispersive part is an even function and the absorptive part an odd function of ω.

We assume that the matrix element of the commutator is a tempered distribution in x, but only derivatives of finite order of the δ functions shall appear. As far as the singularities on the light cone are concerned, the latter property is already guaranteed by the causality condition, which implies that the commutator vanishes for $x^2=x_0{}^2-\mathbf{x}^2<0$. The real polynomial in ω^2 has its origin essentially in the possible appearance of $\epsilon(x_0)$ times $\delta^{(n)}(x^2)$ in the integrand of the dispersive part, which is then not defined for $x=0$. This fact introduces a certain arbitrariness in the Fourier transform, which is just expressed by the polynomial with arbitrary coefficients.

From the assumptions we have made about the matrix element in Eq. (2.5) it is clear that these representations of M_r and M_a are defined only for $\omega^2\geq E_\Delta{}^2$. Following Bogoliubov,[10] we consider therefore the amplitudes as functions of an additional variable β such that we have, for $\omega^2>\beta$,

$$M_{r,a}(\omega,\beta,\Delta^2)=\pm 2E_\Delta i\int d^4x\,\exp[i\omega x_0-i(\omega^2-\beta)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}]$$
$$\times\eta(\pm x_0)\langle p_\Delta|[j(\tfrac{1}{2}x),\,j(-\tfrac{1}{2}x)]|p_{-\Delta}\rangle$$
$$+\sum_{n=0}^{N-1}C_n(\beta,\Delta^2)\omega^{2n}.\quad(2.6)$$

We note that the appearance of $\exp[i(\omega^2-\beta)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}]$ in Eq. (2.6) causes no branch points for $M_{r,a}$ as functions of β, since the symmetry properties of the amplitudes allow its replacement by $\cos[(\omega^2-\beta)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}]$.

Before discussing the analytic properties of $M_{r,a}$ let us explore the absorptive part $A(\omega,\beta,\Delta^2)$. For $\beta<0$ the representation

$$A(\omega,\beta,\Delta^2)=2E_\Delta(\tfrac{1}{2})\int d^4x\,\exp[i\omega x_0-i(\omega^2-\beta)^{\frac{1}{2}}\mathbf{e}\cdot\mathbf{x}]$$
$$\times\langle p_\Delta|[j(\tfrac{1}{2}x),\,j(-\tfrac{1}{2}x)]|p_{-\Delta}\rangle\quad(2.7)$$

is defined for all ω. We assume the existence of a complete set of eigenstates of the energy-momentum operator corresponding to non-negative values of the energy. Then we may decompose the matrix element in Eq. (2.7) with respect to these states. Using translation invariance we find a spectral representation of the form

$$A(\omega,\beta,\Delta^2)=\int_0^\infty d\sigma^2\rho(\sigma^2,\,\beta-\Delta^2,\,\Delta^2)[\delta(\sigma^2-E_\Delta{}^2-\beta$$
$$-2E_\Delta\omega)-\delta(\sigma^2-E_\Delta{}^2-\beta+2E_\Delta\omega)],\quad(2.8)$$

where

$$\rho(\sigma^2,\,\beta-\Delta^2,\,\Delta^2)$$
$$=\pi\textstyle\sum_n(2E_\Delta)^{\frac{1}{2}}\langle p_\Delta|j(0)|\mathbf{p}_n$$
$$=\mathbf{e}[\omega_\beta{}^2(\sigma)-\beta]^{\frac{1}{2}},\,n\rangle(2p_{n0})^{\frac{1}{2}}(2p_{n0})^{\frac{1}{2}}\langle\mathbf{p}_n$$
$$=\mathbf{e}[\omega_\beta{}^2(\sigma)-\beta]^{\frac{1}{2}},\,n|j(0)|p_{-\Delta}\rangle(2E_\Delta)^{\frac{1}{2}}\delta(p_n{}^2-\sigma^2),\quad(2.9)$$

and

$$\omega_\beta(\sigma)=(1/2E_\Delta)(\sigma^2-E_\Delta{}^2-\beta).\quad(2.10)$$

Here σ denotes the "mass" of the system described by $|p_n,n\rangle$, i.e., the total energy in its own rest frame. The summation extends over all possible intermediate states. We have a continuum of intermediate states with $\sigma\geq 2m$ and a discrete one meson state at $\sigma=m$. For reasons of simplicity let us assume here that there are no other discrete states (e.g., bound states) or continua for $\sigma<2m$. More general cases will be discussed in Sec. 3. Under these assumptions about the spectrum, we find that $\rho(\sigma^2,\,\beta-\Delta^2,\,\Delta^2)$ may be written in the form

$$\rho(\sigma^2,\,\beta-\Delta^2,\,\Delta^2)=\pi m^2g^2(\beta-\Delta^2)\delta(\sigma^2-m^2)$$
$$+\Theta(\sigma^2,\,\beta-\Delta^2,\,\Delta^2),\quad(2.11)$$

where $\Theta(\sigma^2,\,\beta-\Delta^2,\,\Delta^2)\equiv 0$ for $\sigma<2m$. On the basis of Lorentz invariance it is easily seen from Eq. (2.9) that ρ must be a real, non-negative quantity. Furthermore, one finds that the function $g^2(\beta-\Delta^2)$, appearing in the contribution from the one-meson state, can *only* depend on $\beta-\Delta^2$.

We see from Eq. (2.6) that for $\beta<0$ and fixed Δ^2 the amplitudes M_r and M_a are analytic functions of ω for $\mathrm{Im}\omega>0$ and $\mathrm{Im}\omega<0$, respectively. This is a consequence of the causality condition. The amplitudes M_r and M_a for real $\omega=\omega_r$ are obtained by the improper limits

$$\lim_{\epsilon\to 0+}M_{r,a}(\omega_r\pm i\epsilon,\,\beta,\,\Delta^2)=M_{r,a}(\omega_r,\beta,\Delta^2).$$

If we take $\beta<-\Delta^2$, we always have a finite gap on the real ω axis where $A(\omega,\beta,\Delta^2)\equiv 0$. Then M_r and M_a are analytic continuations of each other and we have *one* analytic function $M(\omega,\beta,\Delta^2)$, which is regular in the cut ω plane except for a pair of poles at $\omega=\pm\omega_\beta(m)$. The cuts run from $\pm\omega_\beta(2m)$ to $\pm\infty$. From the assumptions we have made about the matrix element of the commutator it follows that for $\omega\to\infty$ $(\mathrm{Im}\omega\neq 0)$, $M(\omega,\beta,\Delta^2)$ does not increase stronger than a certain polynomial. Suppose it vanishes in that limit like $1/\omega$. Then we have for $\beta<-\Delta^2$, $\mathrm{Im}\omega\neq 0$, using the Cauchy theorem and Eqs. (2.4), (2.8), and (2.11),

---

[10] Bogoliubov *et al.*, reference 2.

$$
\begin{aligned}
M(\omega,\beta,\Delta^2)
&=\frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega'\,\frac{A(\omega',\beta,\Delta^2)}{\omega'-\omega}\\
&=\frac{1}{\pi}\int_{\omega_\beta(2m)+\beta/2E_\Delta}^{\infty} d\omega'\,\Theta(2E_\Delta\omega'+E_\Delta{}^2,\beta-\Delta^2,\Delta^2)\\
&\quad\times\left[\frac{1}{\omega'-\omega-\beta/2E_\Delta}+\frac{1}{\omega'+\omega-\beta/2E_\Delta}\right]\\
&\quad+g^2(\beta-\Delta^2)\frac{h(\beta,\Delta^2)}{\omega_\beta{}^2(m)-\omega^2},
\end{aligned}
\tag{2.12}
$$

where $h(\beta,\Delta^2)\equiv -m^2(\beta+\Delta^2)/(4E_\Delta{}^2)$. Note that the lower limit of the integral is independent of β. In the general case we have to supply sufficient powers of ω' in the denominator, which leads to the appearance of an additional polynomial in Eq. (2.12). Since these complications do not cause any principal difficulties in the proof of dispersion formulas, we shall not consider them in detail.

We now wish to continue both sides of Eq. (2.12) from $\beta<-\Delta^2$ to $\beta=E_\Delta{}^2$. To do this we must know the analytic properties of $\Theta(\sigma^2,\beta-\Delta^2,\Delta^2)$ as a function of β. In Sec. 3 we will prove that there exists a $\delta>0$ such that for all $\sigma^2\geq(2m)^2$ and $\Delta^2<\Delta_{\max}{}^2$, $\Theta(\sigma^2,\beta-\Delta^2,\Delta^2)$ is an analytic function of β for $\beta\epsilon S$ (by the notation $a\epsilon A$ we mean that the point a belongs to the set A), where S is the strip

$$S=[\beta:|\mathrm{Im}\beta|<\delta,\ -R<\mathrm{Re}\beta<E_\Delta{}^2+\delta];\tag{2.13}$$

(by "$[a:\cdots]$" we denote "the set of all a which satisfy the condition$\cdots$"), R is any positive number, and δ may go to zero as $R\to\infty$; for all $\beta\epsilon S$ we have $\Theta\equiv0$ for $\sigma^2<(2m)^2$. We know that Θ is a real function of β for real $\beta=\beta_r<0$. Since Θ is analytic in β for $\beta\epsilon S$, it must then also be real for real $\beta=\beta_r\leq E_\Delta{}^2$, $\Delta^2<\Delta_{\max}{}^2$. The limitation on Δ^2 will be discussed in the next section. *In the following we assume only that* $\Delta_{\max}{}^2$ *is finite.*

Using these properties of Θ, we proceed to show that the function $g^2(\beta-\Delta^2)$ must be analytic for $\beta\epsilon S$. Let us denote the integral in Eq. (2.12) by $I(\omega,\beta,\Delta^2)$. If we write Eq. (2.12) in the form

$$[\omega_\beta{}^2(m)-\omega^2][M(\omega,\beta,\Delta^2)-I(\omega,\beta,\Delta^2)]=g^2(\beta-\Delta^2)h(\beta,\Delta^2),\tag{2.14}$$

then the left-hand side is an analytic function of the two complex variables ω and β for $(\omega,\beta)\epsilon D_1$ with

$$D_1=[(\omega,\beta):|\mathrm{Im}\omega|>|\mathrm{Im}(\omega^2-\beta)^{\frac12}|,\quad 2E_\Delta|\mathrm{Im}\omega|>|\mathrm{Im}\beta|,\ \beta\epsilon S].\tag{2.15}$$

The domain D_1 is the intersection of regions in which $M(\omega,\beta,\Delta^2)$ and $I(\omega,\beta,\Delta^2)$ are analytic functions of ω and β. First we prove that $g^2(\beta-\Delta^2)$ is analytic for $\mathrm{Im}\beta\neq0$, $\beta\epsilon S$, by constructing for each of these points β an ω such that $(\omega,\beta)\epsilon D_1$. We write $\omega=\omega_r+i\omega_i$, $\beta=\beta_r+i\beta_i$, and take $\omega_i\neq0$, $|\omega_i|>|\beta_i|/2E_\Delta$, and $\omega_r>\omega_i\beta_r/\beta_i+\beta_i/4\omega_i$, which is evidently always possible and guarantees that $(\omega,\beta)\epsilon D_1$ for $\beta_i\neq0$, $\beta\epsilon S$. Hence $g^2(\beta-\Delta^2)$ is analytic for $\beta\epsilon S$ except for a possible cut on the real β axis. In order to show that there is actually no such cut, we prove that

$$\lim_{\epsilon\to0+}[g^2(\beta_r+i\epsilon-\Delta^2)-g^2(\beta_r-i\epsilon-\Delta^2)]=0\tag{2.16}$$

for $\beta_r\pm i\epsilon=\beta_\pm\epsilon S$. Let us define $\omega_\pm$ by $\omega_\pm=\omega_r\pm i\epsilon/E_\Delta$. Then $(\omega_\pm,\beta_\pm)\epsilon D_1$ provided we choose $\omega_r>(5/4)E_\Delta+\delta/E_\Delta$ and $\epsilon>0$ and sufficiently small. But for $(\omega,\beta)\epsilon D_1$ we may use the representation (2.6) for $M(\omega,\beta,\Delta^2)$ and find, recalling Eq. (2.4),

$$\lim_{\epsilon\to0+}[M(\omega_+,\beta_+,\Delta^2)-M(\omega_-,\beta_-,\Delta^2)]=2iA(\omega_r,\beta_r,\Delta^2);\tag{2.17}$$

in addition it follows from Eq. (2.12) that

$$
\begin{aligned}
\lim_{\epsilon\to0+}[I(\omega_+,\beta_+,\Delta^2)-I(\omega_-,\beta_-,\Delta^2)]
&=2i[\Theta(2E_\Delta\omega_r+E_\Delta{}^2+\beta_r,\beta_r-\Delta^2,\Delta^2)\\
&\quad-\Theta(-2E_\Delta\omega_r+E_\Delta{}^2+\beta_r,\beta_r-\Delta^2,\Delta^2)].
\end{aligned}
\tag{2.18}
$$

Evidently Eqs. (2.14), (2.17), and (2.18) imply Eq. (2.16). We conclude that $g^2(\beta-\Delta^2)$ may be continued analytically onto the domain S. Since we know from Eq. (2.9) that $g^2(\beta-\Delta^2)$ is real for $\beta=\beta_r<0$, the same must be true for all $\beta\epsilon S$ with $\mathrm{Im}\beta=0$.

If we relax the assumption about the behavior of M for $\omega\to\infty$, we will have on the right-hand side of Eq. (2.14) in addition to the g^2 term a polynomial of the form

$$\sum_{n=0}^{N-1}C_n(\beta,\Delta^2)\omega^{2n}.$$

The proof that the coefficients C_n are analytic functions of β for $\beta\epsilon S$ is completely equivalent to the proof of the corresponding properties for $g^2(\beta-\Delta^2)$.

For the application of dispersion relations it is important to show that $g^2(m)$ is non-negative. In order to prove this we consider the invariant matrix element

$$(2k_0)^{\frac12}\langle k|j(0)|p\rangle(2p_0)^{\frac12}=m\Gamma[(p-k)^2],\quad(k^2=p^2=m^2),\tag{2.19}$$

in a system where $\mathbf{p}=\mathbf{0}$. From the representation

$$
\begin{aligned}
\Gamma[2m(m-k_0)]&=-i\left(\frac{2}{m}\right)^{\frac12}\int d^4x\,\exp(ik\cdot x)\eta(x_0)\\
&\quad\times\langle0|[j(x),j(0)]|0,m\rangle+\sum_{n=0}^{N-1}a_nk_0{}^n,
\end{aligned}
\tag{2.20}
$$

and our assumptions about the spectrum one can easily see, using the methods described earlier by one of us (R. O.),[11] that Γ is an analytic function of $\lambda=2m(m-k_0)$ in the cut λ plane, the cut runs from $(2m)^2$ to $+\infty$. Then the function $\Gamma(\lambda)$ is real for $\lambda_i=0$, $\lambda_r<(2m)^2$, since it is for $\lambda_r<0$. Evidently $\Gamma^2(\lambda)$ has the same properties; in addition it is non-negative for $\lambda=\lambda_r<(2m)^2$. We see from Eqs. (2.9), (2.10), and (2.11) that $g^2(\beta-\Delta^2)=\Gamma^2(\lambda)$ with $\lambda=\beta-\Delta^2$. Hence we have shown that $g^2(m)\geq 0$. We could have used the properties of $\Gamma(\lambda)$ mentioned above in order to demonstrate that $g^2(\beta-\Delta^2)$ is analytic for $\beta\epsilon S$. However, the method described earlier is still useful in order to prove the analyticity of the coefficients $C_n(\beta,\Delta^2)$ of the polynomial. Let us now go back to Eq. (2.12). Since Θ and g^2 are analytic functions of β for $\beta\epsilon S$ we realize that the right-hand side of this equation is analytic in ω, β for $(\omega,\beta)\epsilon D_2$, where

$$D_2=[(\omega,\beta):2E_\Delta|\mathrm{Im}\omega|>|\mathrm{Im}\beta|,\beta\epsilon S]. \quad (2.21)$$

Hence we can continue $M(\omega,\beta,\Delta^2)$ into the domain D_2 so as to equal $I(\omega,\beta,\Delta^2)+g^2(\beta-\Delta^2)h(\beta,\Delta^2)[\omega_\beta^2(m)-\omega^2]^{-1}$. But any (ω,β) with $\beta=E_\Delta^2$, $\mathrm{Im}\omega\neq 0$ is contained in D_2 and hence we obtain from Eq. (2.12) the Hilbert relation

$$\begin{aligned}M(\omega,E_\Delta^2,\Delta^2)=&\frac{1}{\pi}\int_{\omega(2m)}^{E_\Delta}d\omega'\Theta(2E_\Delta\omega'+2E_\Delta^2,m^2,\Delta^2)\\&\times\left[\frac{1}{\omega'-\omega}+\frac{1}{\omega'+\omega}\right]+\frac{2}{\pi}\int_{E_\Delta}^{\infty}d\omega'\frac{\omega'A(\omega',\Delta^2)}{\omega'^2-\omega^2}\\&+\frac{g^2(m^2)m^2}{2E_\Delta}\left[\frac{1}{\omega(m)-\omega}+\frac{1}{\omega(m)+\omega}\right],\end{aligned} \quad (2.22)$$

where $\mathrm{Im}\omega\neq 0$, $\omega(\sigma)=\omega_\beta(\sigma)$ for $\beta=E_\Delta^2$ and $A(\omega',\Delta^2)\equiv A(\omega',E_\Delta^2,\Delta^2)$. For the M thus continued, it remains to be proved that the improper limits

$$\lim_{\epsilon\to 0+}M(\omega_r\pm i\epsilon,E_\Delta^2,\Delta^2)=M_{r,a}(\omega_r,\Delta^2)$$

hold, provided $\omega_r>E_\Delta$. For each $\omega_r^2>E_\Delta^2+\epsilon^2$ and $\beta=E_\Delta^2+2ia\epsilon$, we can always find an a such that $\omega_r-(\omega_r^2-E_\Delta^2)^{\frac{1}{2}}<a<E_\Delta$. But then we have, for ϵ small enough, $(\omega_r\pm i\epsilon, E_\Delta^2\pm 2ia\epsilon)\epsilon D_1\subset D_2$ and consequently

$$\begin{aligned}\lim_{\epsilon\to 0+}M(\omega_r\pm i\epsilon,E_\Delta^2,\Delta^2)&=\lim_{\epsilon\to 0+}M(\omega_r\pm i\epsilon,E_\Delta^2\pm i2a\epsilon,\Delta^2)\\&=M_{r,a}(\omega_r,\Delta^2),\end{aligned} \quad (2.23)$$

because for $(\omega,\beta)\epsilon D_1$ the Fourier representations (2.6) for M_r and M_a are valid. (By $B\supset A$ or $A\subset B$ we denote the fact that all points of A belong also to B.)

Equation (2.22) represents the desired dispersion formula for $M(\omega,\Delta^2)$. The function $\Theta(2E_\Delta\omega'+2E_\Delta^2,m^2,\Delta^2)$ appearing on the first integral is the proper extension of the absorptive part on the unphysical region.

3. ANALYTIC CONTINUATION

In this section we wish to prove that for every fixed $\sigma^2\geq(2m)^2$ and $\Delta^2<\Delta_{\max}^2$ the quantity $\Theta(\sigma^2,\beta-\Delta^2,\Delta^2)$ is an analytic function in β for $\beta\epsilon S$, where S is the region defined by Eq. (2.13). Since $\sigma^2\geq(2m)^2$ it is sufficient to show that

$$(\sigma^2-m^2)\Theta(\sigma^2,\beta-\Delta^2,\Delta^2)=(\sigma^2-m^2)\rho(\sigma^2,\beta-\Delta^2,\Delta^2)$$

has the required properties. The quantity ρ is related to $A(\omega,\beta,\Delta^2)$ by

$$A(\omega,\beta,\Delta^2)=\bar{A}(\omega,\beta,\Delta^2)-\bar{A}(-\omega,\beta,\Delta^2), \quad (3.1)$$

with

$$\bar{A}(\omega,\beta,\Delta^2)=\rho(2E_\Delta\omega+E_\Delta^2+\beta,\beta-\Delta^2,\Delta^2).$$

Let us define a function $A(p_1p_2p_3p_4)$ by the equation

$$A(p_1p_2p_3p_4)=A(k+k',p',p), \quad (3.2)$$

where $p_1=p'$, $p_2=-p$, $p_3=k'$, $p_4=-k$. For reasons of simplicity we have not introduced a new symbol in Eq. (3.2). By standard methods[10] we have then the representation

$$\begin{aligned}&(2\pi)^4\delta(p_1+p_2+p_3+p_4)A(p_1p_2p_3p_4)\\&=\tfrac{1}{2}\int dx_1dx_2dx_3dx_4\exp[i(p_1x_1+p_2x_2\\&\qquad+p_3x_3+p_4x_4)]F(x_1x_2x_3x_4),\end{aligned} \quad (3.3)$$

where

$$\begin{aligned}F(x_1x_2x_3x_4)=&\left\langle 0\left|\frac{\delta^2}{\delta\phi(x_1)\delta\phi(x_2)}[j(x_3),j(x_4)]\right|0\right\rangle\\&+\text{degenerate terms.}\end{aligned} \quad (3.4)$$

The operator $\delta/\delta\phi(x)$ is that of a functional derivation with respect to the boson field at the point x, where this field is regarded as a classical one while the derivation is being performed. We use it here only as a convenient shorthand and the expectation value F may be easily written out in terms of commutators and step functions. The "degenerate terms" in Eq. (3.4) contain equal time commutators; they do not alter the properties of $A(p_1\cdots p_4)$ in which we are interested, and therefore we do not need to consider them explicitly.

Carrying out the functional derivations in Eq. (3.4), we obtain

$$F(x_1x_2x_3x_4)=\bar{F}(x_1x_2x_3x_4)-\bar{F}(x_1x_2x_4x_3),$$

and correspondingly

$$A(p_1p_2p_3p_4)=\bar{A}(p_1p_2p_3p_4)-\bar{A}(p_1p_2p_4p_3),$$

where

$$\begin{aligned}F(x_1x_2x_3x_4)=&\left\langle 0\left|\left[\frac{\delta j(x_3)}{\delta\phi(x_1)},\frac{\delta j(x_4)}{\delta\phi(x_2)}\right]\right|0\right\rangle\\&+\left\langle 0\left|\left[j(x_3),\frac{\delta^2 j(x_4)}{\delta\phi(x_1)\delta\phi(x_2)}\right]\right|0\right\rangle.\end{aligned} \quad (3.5)$$

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[11] Reinhard Oehme, Nuovo cimento 10, 1316 (1956).

The condition on the spectrum, which we have discussed in Sec. 2, implies that the Fourier transform of $\langle 0|[j(x_3),\delta^2 j(x_4)/\delta\phi(x_1)\delta\phi(x_2)]|0\rangle$ is zero for $p_3^2<(2m)^2$. Since we are only concerned with the region $p_3^2<(2m)^2$, $p_4^2<(2m)^2$ throughout this paper, we may, without error, take as zero the second term on the right-hand side of Eq. (3.5).

An expression for $j(x)$, equivalent to that used in the last section, is

$$j(x)=i\frac{\delta S}{\delta\phi(x)}S^{+}.$$

The causality condition may now be written as

$$\delta j(x)/\delta\phi(y)=-i\eta(y_0-x_0)[j(y),j(x)]=0 \qquad \text{for} \quad (y_0-x_0)<|\mathbf{y}-\mathbf{x}|. \tag{3.6}$$

From the causality condition it follows that $\bar{F}(x_1\cdots x_4)$ is retarded in the variables (x_1-x_3) and (x_2-x_4), and so will be denoted by $\bar{F}_{rr}(x_1\cdots x_4)$. The four functions which are obtained from $\bar{F}_{rr}(x_1\cdots x_4)$ by interchange of the variables x_1 and x_3, x_2 and x_4 any number of times will be denoted ky $\bar{F}_{ij}(x_1x_2x_3x_4)$, with i, $j=r$, a. The subscripts i, j correspond to the advanced or retarded property in the variables (x_1-x_3), (x_2-x_4), respectively. For example, we have

$$\bar{F}_{ar}(x_1x_2x_3x_4)=\left\langle 0\left|\left[\frac{\delta j(x_1)}{\delta\phi(x_3)},\frac{\delta j(x_4)}{\delta\phi(x_2)}\right]\right|0\right\rangle.$$

By the use of the relation

$$\frac{\delta j(x)}{\delta\phi(y)}-\frac{\delta j(y)}{\delta\phi(x)}=i[j(x),j(y)],$$

and the condition on the spectrum, we obtain

$$\begin{aligned} &\bar{A}_{rj}(p_1\cdots p_4)-\bar{A}_{aj}(p_1\cdots p_4)=0 \\ &\qquad\qquad \text{for} \quad p_1^2<(2m)^2, \quad p_3^2<(2m)^2 \\ &\bar{A}_{ir}(p_1\cdots p_4)-\bar{A}_{ia}(p_1\cdots p_4)=0 \\ &\qquad\qquad \text{for} \quad p_2^2<(2m)^2, \quad p_4^2<(2m)^2. \end{aligned} \tag{3.7}$$

Also, by the same condition,

$$[(p_1+p_3)^2-m^2]\bar{A}_{ij}(p_1\cdots p_4)=0 \tag{3.8}$$

for $(p_1+p_3)^2<(2m)^2$ or $p_{10}+p_{30}<2m$, and all i, $j=r$, a.

We consider from now on only the function $\bar{B}_{ij}$, $\bar{G}_{ij}$ which are defined by the expressions

$$\bar{B}_{ij}(p_1\cdots p_4)=[(p_1+p_3)^2-m^2]\bar{A}_{ij}(p_1\cdots p_4), \tag{3.9}$$

$$\bar{G}_{ij}(x_1\cdots x_4) =\left[-\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_3}\right)^2-m^2\right]\bar{F}_{ij}(x_1\cdots x_4). \tag{3.10}$$

The analyticity property of $\Theta(\sigma^2,\beta-\Delta^2,\Delta^2)$ which we desire is then proved by Theorem 1[3] (for $\Delta_{\max}^2=m^2$).

Theorem 1.—We are given four generalized functions of the four-vector variables $x_1, \cdots, x_4$,

$$\tilde{f}_{ij}(x_1x_2x_3x_4), \quad i=r,a, \quad j=r,a.$$

The $\tilde{f}_{ij}$ are assumed to be tempered distributions, multiplied by certain step functions, so that their Fourier transforms are defined to within certain arbitrary polynomials. The quantities f_{ij} are assumed invariant under the transformations of the inhomogeneous, orthochronous Lorentz group, and are retarded or advanced in (x_1-x_3), (x_2-x_4) as denoted by the subscripts i, j. The Fourier transforms of the f_{ij}, defined by the expressions

$$\begin{aligned} &(2\pi)^4 f_{ij}(p_1p_2p_3p_4)\delta(p_1+p_2+p_3+p_4) \\ &\quad=\int dx_1dx_2dx_3dx_4 \exp[i(p_1x_1+p_2x_2 \\ &\qquad\qquad +p_3x_3+p_4x_4)]\tilde{f}_{ij}(x_1x_2x_3x_4), \end{aligned}$$

are assumed to have the properties:

$$\begin{aligned} &f_{rj}-f_{aj}\equiv 0 \text{ for } p_1^2<(2m)^2, \text{ and } p_3^2<(2m)^2, \\ &\qquad\qquad\qquad \text{and } j=r \text{ or } a; \\ &f_{ir}-f_{ia}\equiv 0 \text{ for } p_2^2<(2m)^2, \text{ and } p_4^2<(2m)^2, \\ &\qquad\qquad\qquad \text{and } i=r \text{ or } a; \\ &f_{ij}\equiv 0 \text{ for } (p_1+p_3)^2<(2m)^2 \text{ or } (p_{10}+p_{30})<0, \\ &\qquad\qquad\qquad i=r \text{ or } a, \quad j=r \text{ or } a. \end{aligned} \tag{3.11}$$

Then we wish to prove that there is a function $\chi(z_1,z_2,z_3,z_4,z_5;z_6)$ of the complex variables $z_1, \cdots, z_5$ and the real variable z_6, which is in general a tempered distribution in z_6 and which has the following properties: (1) For each real z_6, $\chi(z_1,\cdots,z_6)$ is analytic in $z_1, \cdots, z_5$ in the region D,

$$\begin{aligned} D=[z_1, \cdots, z_5: &|z_1-m^2|<\delta, |z_2-m^2|<\delta, \\ &|z_3-\gamma|<\delta, |z_4-\gamma|<\delta, |z_5+4\Delta^2|<\delta], \end{aligned} \tag{3.12}$$

where $-R\leq\gamma M m^2$, for any positive number R, and where δ is some small positive number, which may become zero as $R\rightarrow\infty$. We require also $\Delta^2<m^2$. (2) For $p_1, \cdots, p_4$ real, $p_1+p_2+p_3+p_4=0$, $p_{10}+p_{30}>0$, $z_1=p_1^2$, $z_2=p_2^2$, $z_3=p_3^2$, $z_4=p_4^2$, $z_5=(p_1+p_2)^2$, and $z_6=(p_1+p_3)^2$, with $(z_1,\cdots,z_5)\epsilon D$, we have the representation

$$f_{ij}(p_1p_2p_3p_4)=\chi(z_1,z_2,z_3,z_4,z_5;z_6).$$

(3) $\chi\equiv 0$ for $z_6<(2m)^2$.

Proof.—For convenience we introduce the independent four vectors q_1, q_2, and q_3, by

$$p_1=q_1+q_3, \quad p_2=-q_2-q_3, \quad p_3=-q_1+q_3, \quad p_4=q_2-q_3.$$

Then, writing $g_{ij}(q_1q_2q_3)=f_{ij'}(p_1p_2p_3p_4)$, where for $j=r$ then $j'=a$, and for $j=a$ then $j'=r$, we have

$$\begin{aligned} g_{ij}(q_1q_2q_3)=\int dy_1dy_2dy_3 &\exp[i(q_1y_1 \\ &+q_2y_2+q_3y_3)]\tilde{g}_{ij}(y_1y_2y_3), \end{aligned} \tag{3.13}$$

where

$$\tilde{f}_{ij'}(x_1x_2x_3x_4)=\tilde{g}_{ij}(y_1y_2y_3)$$

at $y_1=x_1-x_3$, $y_2=-(x_2-x_4)$, $y_3=(x_1-x_2+x_3-x_4)$.

From Eq. (3.13) and the retarded and advanced character of the g_{ij} we see that the functions g_{ij} are analytic in certain regions. If we take these functions as one function $g(q_1q_2q_3)$, then $g(q_1q_2q_3)$ is analytic in the region $(q_1,q_2)\epsilon W\times W$ for each real q_3, where

$$W=[q\colon |\mathrm{Im}q_0|>|\mathrm{Im}\mathbf{q}|,\ |\mathrm{Re}q_0|<\infty,\ |\mathrm{Re}\mathbf{q}|<\infty].$$

[Note that $q_3=\frac{1}{2}(p_1+p_3)$ is always real.]

Let S be a set in the four-dimensional real space R_4, which is defined by

$$S=[q\colon \mathrm{Im}q=0,\ (\mathrm{Re}q+q_3)^2<(2m)^2,\ (\mathrm{Re}q-q_3)^2<(2m)^2]$$

for each q_3. Then the equality properties between the various g_{ij} for real q_1, q_2, and q_1 or q_2 in S, which correspond to Eq. (3.11) for the f_{ij}, may be immediately extended to equalities satisfied for q_1 in S and q_2 in $W\cup R_4$ or vice versa. (By $A\cup B$ we denote the union of the sets A and B.) Explicitly, we have

$$\begin{aligned}
&g_{rj}(q_1q_2q_3)-g_{aj}(q_1q_2q_3)\equiv 0\\
&\qquad\qquad \text{for}\quad q_1\epsilon S\quad\text{and}\quad q_2\epsilon W\cup R_4,\\
&g_{ir}(q_1q_2q_3)-g_{ia}(q_1q_2q_3)\equiv 0\\
&\qquad\qquad \text{for}\quad q_2\epsilon S\quad\text{and}\quad q_1\epsilon W\cup R_4,\\
&g_{ij}(q_1q_2q_3)\equiv 0\quad\text{for}\quad q_3{}^2<m^2,\\
&\qquad\qquad (q_1,q_2)\epsilon(W\cup R_4)\times(W\cup R_4).
\end{aligned}\tag{3.14}$$

We thus see, that, for fixed q_3, the function $g(q_1q_2q_3)$ satisfies the conditions of the edge of the wedge theorem, which we have formulated in the Appendix. Hence by this theorem, we may continue $g(q_1q_2q_3)$ in (q_1,q_2) to be analytic in the region $(W\cup N)\times(W\cup N)$, where N is some neighborhood of the set S.

Since $q_3{}^2\geq m^2$, we may now, without loss of generality, choose a frame of reference in which $\mathbf{q}_3=0$ and write $q_{30}=t$. Then, for a given t, g is a function of q_{10}, q_{20}, and the two 3-vectors $\mathbf{q}_1$, $\mathbf{q}_2$. With respect to $\mathbf{q}_1$ and $\mathbf{q}_2$ it is invariant under the transformations of the orthogonal group and analytic in a region which is also invariant. One can prove that g depends only upon the inner products $\mathbf{q}_1{}^2$, $\mathbf{q}_2{}^2$, $\mathbf{q}_1\cdot\mathbf{q}_2$, and of course upon q_{10} and q_{20}. It will be analytic in these variables in the domain corresponding to $(q_1,q_2)\epsilon(W\cup N)\times(W\cup N)$. The proof in question makes use of the compactness of the orthogonal group and invariance of the Haar measure.[12] Instead of the variables q_{10}, q_{20}, $\mathbf{q}_1{}^2$, $\mathbf{q}_2{}^2$, $\mathbf{q}_1\cdot\mathbf{q}_2$ we choose to consider the variables z_1, z_2, z_3, z_4, z_5, which are defined by the equations

$$z_1=(q_1+q_3)^2,\ z_2=(q_2+q_3)^2,\ z_3=(q_1-q_3)^2,\ z_4=(q_2-q_3)^2,$$
$$z_5=(q_1-q_2)^2,\ z_6=4q_3{}^2\ (q_{30}=t,\ \mathbf{q}_3=0),$$

and are related to the above-mentioned variables by a simple analytic transformation. We write $g=\chi(z_1,\cdots,z_5;z_6)$, where χ is analytic in $z_1,\ \cdots,\ z_5$, for each fixed, real z_6, in the domain corresponding to $(W\cup N)\times(W\cup N)$.

We are interested in the behavior of χ near the points

$$z_1=m^2,\ z_2=m^2,\ z_3=\gamma,\ z_4=\gamma,\ z_5=-4\Delta^2,\ z_6=4t^2,\tag{3.15}$$

where $-R<\gamma\leq m^2$ and $t\geq m$. At these values of $z_1,\ \cdots,\ z_6$ we have

$$\begin{aligned}
q_{10}&=q_{20}=(m^2-\gamma)/4t,\\
\mathbf{q}_1&=\rho(t,\gamma)\mathbf{e}_1+\Delta\mathbf{e}_2,\\
\mathbf{q}_2&=\rho(t,\gamma)\mathbf{e}_1-\Delta\mathbf{e}_2,
\end{aligned}\tag{3.16}$$

where $\mathbf{e}_1\cdot\mathbf{e}_2=0$ and $\rho^2(t,\gamma)=[t+(m^2-\gamma)/4t]^2-E_\Delta{}^2$. We wish to show that for $\gamma\leq m^2$, for each $t\geq m$ and $\Delta^2<m^2$, the values of (q_1,q_2) lie in the region of analyticity of $g(q_1q_2q_3)$. This will show that the corresponding value of $z_1,\ \cdots,\ z_5$ lies in the region of analyticity of $\chi(z_1,\cdots,z_5;z_6)$. Let us first take a small $\bar{\delta}>0$ and assume that $\Delta^2<\bar{\delta}^2$. Then we see from Eq. (3.16) that for $\gamma\leq m^2$, $t\geq m$ the inequalities $|\mathrm{Im}\mathbf{q}_i|<\bar{\delta}$, $(\mathrm{Re}q_{i0}\pm t)^2-\mathbf{q}_i{}^2\leq m^2$ hold with $i=1,2$, and if $\bar{\delta}$ is sufficiently small $(\bar{\delta}<\delta)$ the 4-vectors q_1 and q_2 both lie in N. We see that the edge of the wedge theorem is sufficient to prove the theorem for $0\leq\Delta^2<\bar{\delta}^2$ and hence the dispersion relations for forward and near-forward scattering (derivative amplitudes).[13]

For general Δ^2 it is true that *all* relevant, corresponding values of (q_1,q_2) lie in N for each $t\geq E_\Delta$; but this is no longer the case for $t<E_\Delta$. Hence we need to extend the region of analyticity of g. To consider this in more detail we have drawn in Fig. 1 the curve (III) for

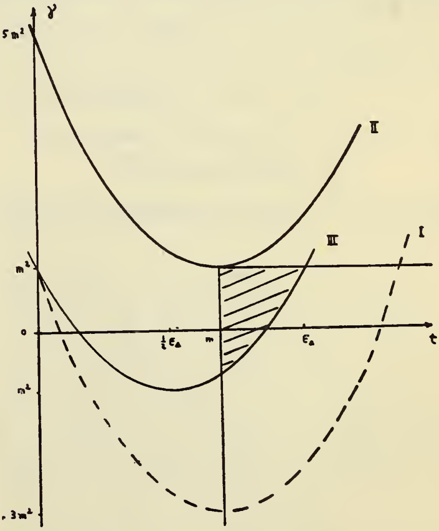

FIG. 1. This figure is drawn for $\Delta^2=m^2$. The region of interest is given by $t\geq m$, $\gamma\leq m^2$. The corresponding values of (q_1,q_2) lie in the region of analyticity given by the semitube method (shaded region) or the edge of the wedge theorem (unshaded region).

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[12] We are indebted to Professor L. Ehrenpreis for discussions concerning this proof.

[13] K. Symanzik, Phys. Rev. 105, 743 (1957). This paper contains further references.

$\rho^2(t,\gamma)=0$. In the region defined by $\rho^2(t,\gamma)>-\bar{\delta}^2$, which is given by

$$[t-\tfrac{1}{2}(E_\Delta{}^2-\bar{\delta}^2)^{\frac{1}{2}}]^2>\tfrac{1}{4}(\Delta^2-\bar{\delta}^2+\gamma),$$

we always have the corresponding values of q_1, q_2 lying in N, while for other values of t, γ this is not true. This latter region is completely contained within the region $m\leq t<E_\Delta$, $\gamma\leq m^2$.

The ideal solution of the problem would be to obtain the envelope of holomorphy of the domain $[W\cup D(t,\Delta^2)\times W\cup D(t,\Delta^2)]$, where

$$D(t,\Delta^2)=[q:\mathrm{Im}q=0,\ (\mathrm{Re}q_0+t)^2-(\mathrm{Re}\mathbf{q})^2<4m^2,$$
$$(\mathrm{Re}q_0-t)^2-(\mathrm{Re}\mathbf{q})^2<4m^2,\ m\leq t<E_\Delta].$$

There are several methods of obtaining this region of holomorphy, for example by means of the continuity theorem,[14] or by the general method of Bremermann[7]; we shall not attempt to solve here the general problem.

For $\Delta^2<3m^2$ the region $D(t,\Delta^2)$ is as in Fig. 2, while for $\Delta^2>3m^2$ we also have the other topological possibility for $D(t,\Delta^2)$, as shown in Fig. 3. We shall restrict ourselves here to $\Delta^2<3m^2$. Then, for $m\leq t<E_\Delta$, we have

$$D(t,\Delta^2)\supset D'=[q:\mathrm{Im}q=0,\ |\mathrm{Re}q_0|<\eta(t),\ |\mathrm{Re}\mathbf{q}|<\infty],$$

where $\eta(t)=(2m-t)$. For this case we apply lemma 3 of the Appendix to $g(q_1,q_2,q_3)$, where E of that theorem is now the subspace D' of R_4. Then the lemma shows that $g(q_1,q_2,q_3)$ is analytic if $(q_1,q_2)\epsilon(W\cup N')\times(W\cup N')$ for each $q_3{}^2=t^2$, and $m\leq t<E_\Delta$, where N' is a semitube neighborhood of D'.

We now write $(W\cup N')$ to exhibit its property of being a semitube,

$$W\cup N'=[q:q_0\epsilon B,\ |\mathrm{Im}\mathbf{q}|<v(q_0),\ |\mathrm{Re}\mathbf{q}|<\infty],$$

where $B=[q_0\text{-plane}]-[q_0:\mathrm{Im}q_0=0,\ |\mathrm{Re}q_0|\geq\eta(t)]$

$$v(q_0)=\sup[|\mathrm{Im}q_0|,v^*(q_0)]$$

and $v^*(q_0)=\frac{1}{4}[\eta(t)-|\mathrm{Re}q_0|]$, according to Lemma 3 of the Appendix.

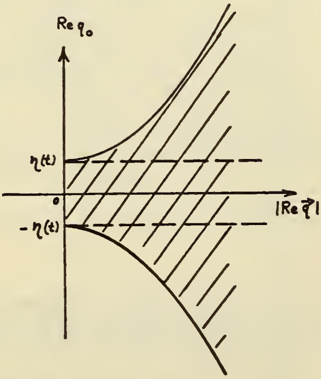

FIG. 2. The set $D(t,\Delta^2)$ for $\Delta^2<3m^2$.

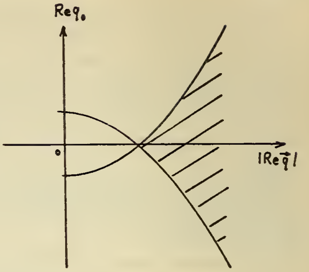

FIG. 3. The set $D(t,\Delta^2)$ for $\Delta^2>3m^2$, $t>2m$.

One can prove on the basis of the continuity theorem that the envelope of holomorphy of this semitube will be

$$E(W\cup N')=[q:q_0\epsilon B,\ |\mathrm{Im}\mathbf{q}|<V(q_0),\ |\mathrm{Re}\mathbf{q}|<\infty],$$

where $V(q_0)$ is the *smallest* superharmonic majorant of $v(q_0)$.[6]

We now prove that $V(q_0)=\tilde{V}(q_0)$, where $\tilde{V}(q_0)$ is defined to be the function $\mathrm{Im}[q_0{}^2-\eta^2(t)]^{\frac{1}{2}}$. For $\tilde{V}(q_0)$ is harmonic in B, and it may be seen by a straightforward calculation that $\tilde{V}(q_0)\geq v(q_0)$. Hence $\tilde{V}(q_0)$ is a superharmonic majorant of $v(q_0)$, and so

$$\tilde{V}(q_0)\geq V(q_0).$$

To obtain the reverse inequality, we note that $\tilde{V}(q_0)$ satisfies the same boundary conditions on the cut and asymptotically for $q_0\to\infty$ as does $v(q_0)$, and we have $V(q_0)\geq v(q_0)$. Since the Dirichlet problem for such a "boundary" has a unique solution, then this is $\tilde{V}(q_0)$, and hence by the definition of superharmonic functions[15] $V(q_0)\geq\tilde{V}(q_0)$ for all $q_0\epsilon B$.

The envelope of holomorphy of $(W\cup N')\times(W\cup N')$ is then $E(W\cup N')\times E(W\cup N')$, which we shall denote by H. At the points of interest [Eq. (3.16)] we have q_{10}, $q_{20}\epsilon B$, provided

$$(m^2-\gamma)/4t<\eta(t). \qquad (3.17)$$

The curve of equality in (3.17) is drawn in Fig. 1, the required region being inside the parabola I.

Then the points of interest lie in H if $|\mathrm{Im}\mathbf{q}_i|<|\mathrm{Im}(q_{i0}{}^2-\eta^2(t))^{\frac{1}{2}}|$, $(i=1,2)$, which implies

$$|\mathrm{Im}\rho(t,\gamma)|<\left|\mathrm{Im}\left[\left(\frac{m^2-\gamma}{4t}\right)^2-\eta^2(t)\right]^{\frac{1}{2}}\right|. \qquad (3.18)$$

The curve of equality of (3.18) is also drawn in Fig. 1, the required region being outside the parabola II.

In the region $t\geq m$, $\gamma\leq m^2$ we see from these curves that for $\Delta^2<m^2$ the part of this region not covered by the edge of the wedge theorem [the part for which $\rho(t,\gamma)$ is pure imaginary] is now contained in the region inside I and outside II. Hence the corresponding points q_1, q_2 lie inside H.

This proves our Theorem 1.

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[14] See reference 4, p. 49, Satz 17; also reference 6.

[15] For the definition and discussion of superharmonic functions see F. Riesz, Acta Math. 48, 329 (1926); also references 4 and 6.

4. EXTENSION TO OTHER CASES

Our discussion in the previous two sections has given a proof of the dispersion relations for the scattering of bosons with equal mass, provided $\Delta^2 < m^2$. We know that this specific restriction is merely due to our limited analytic continuation, because, by employing the semitube method, we have not made sufficient use of those regions of the domain E, which are important for larger values of Δ^2. By the methods of Bremermann[7,16] it is possible to make better use of E, and further calculations show that we can obtain $\Delta_{\max} = 2m^2$.

Let us now discuss how the proof may be extended to other cases of more physical interest. One can easily see that the essential changes are due to different spectral conditions. Therefore it is useful to formulate the assumptions of Theorem 1 in a more general form using adjustable parameters.

Instead of the conditions (3.11), we now require that the Fourier transforms $f_{ij}(p_1,p_2,p_3,p_4)$ satisfy the following relations:

$$\begin{aligned} f_{rj}-f_{aj}&\equiv 0 \quad \text{for} \quad p_1^2<a^2 \quad \text{and} \quad p_3^2<b^2, \quad j=r,a;\\ f_{ir}-f_{ia}&\equiv 0 \quad \text{for} \quad p_2^2<a^2 \quad \text{and} \quad p_4^2<b^2, \quad i=r,a;\\ f_{ij}&\equiv 0 \quad \text{for} \quad (p_1+p_3)^2<\kappa^2 \quad \text{or} \quad p_{10}+p_{30}<0,\\ &\qquad\qquad i=r\ a; \quad j=r,a. \end{aligned} \tag{4.1}$$

Then there is a function $\chi(z_1,z_2,z_3,z_4,z_5;z_6)$ with the properties (1), (2), and (3) of Theorem 1, except that we are now interested in values of γ, with $-R<\gamma \leq \gamma_{\text{phys}}$. These values of γ give values of q_1, q_2 which are in the region of analyticity given by the semitube method, provided $\Delta^2 < \Delta_{\max}^2$. Furthermore, we have in (3) $\chi \equiv 0$ for $z_6 < \kappa^2$.

First we consider the *dispersion relations for pion-nucleon scattering*. The discussions of Secs. 2 and 3 can be readily applied to this case, and from the spectral conditions we find

$$a=m+\mu, \quad b=3\mu, \quad \kappa^2=(m+\mu)^2.$$

The physical value for γ is $\gamma_{\text{phys}}=\mu^2$, and by means of the edge of the wedge and the semitube theorems we find $\Delta_{\max}^2 = 2\mu^2$. *Since there are no other restrictions on* Δ^2 *than those connected with the analytic continuation in the proof of Theorem 1*, we have given a derivation of the pion-nucleon dispersion formulas for $\Delta^2 < 2\mu^2$. As we have discussed in the equal-mass case, this limit may be removed by further analytic continuation.§

We turn now to the problem of deriving *dispersion relations for nucleon-nucleon scattering*. Again the considerations of Secs. 2 and 3 go through straightforwardly and a brief discussion corresponding to those of Sec. 1 has been given by Goldberger, Nambu, and one of us (R. O.).[1] For the parameters in Theorem 1, we have

$$a=b=m+\mu, \quad \kappa^2=(2\mu)^2, \quad \gamma_{\text{phys}}=m^2.$$

One finds, that even for $\Delta^2=0$ the points (q_1,q_2) with (γ,t) values

$$\gamma=m^2, \ \tfrac{1}{2}\{(m+\mu)-[(m-\mu)^2-2\mu^2]^{\frac{1}{2}}\}<t \\ <\tfrac{1}{2}\{(m+\mu)+[(m-\mu)^2-2\mu^2]^{\frac{1}{2}}\} \tag{4.2}$$

[q_1, q_2 are given in forms of γ, t by Eq. (3.16)], do not lie in the region of analytic continuation of the absorptive part. Since the function χ becomes zero only for $t<\frac{1}{2}\kappa=\mu$, we cannot assure the validity of the nucleon-nucleon relations. The troublesome region of t given in Eq. (4.2) vanishes only for

$$\mu > (\sqrt{2}-1)m, \tag{4.3}$$

which is much larger than the experimental mass ratio. The limitation (4.3) is *not* due to our use of the semitube method, because one can easily give examples of functions $g(q_1,q_2,q_3)=\chi(z_1,z_2,z_3,z_4,z_5;z_6)$ which are analytic in z_1, $\cdots$, z_5 in the region corresponding to $(q_1,q_2)\epsilon[W\cup D(t,0)]\times[W\cup D(t,0)]$, but have singularities at the points $z_1=z_2=m^2$, $z_3=z_4=\gamma$ and $z_5=0$ if $\gamma=2(m+\mu-t)^2+2t^2-m^2$. For $\gamma=m^2$ this gives $t=\frac{1}{2}\{(m+\mu)\pm[(m-\mu)^2-2\mu^2]^{\frac{1}{2}}\}$, which is in the range $t\geq\mu$. The functions[17]

$$g(q_1,q_2,t)=\frac{\rho(t)}{[\alpha^2-q_1^2]^n[\alpha^2-q_2^2]^n}, \tag{4.4}$$

with $\rho(t)=0$ for $t<\mu$, $\rho(t)>0$ for $t\geq\mu$, $n\geq 1$ and $\alpha^2 =(m+\mu-t)^2$ have such properties. For the corre-

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[16] Hans J. Bremermann, Trans. Am. Math. Soc. 82, 17 (1956).

§ *Note added in proof.*—We have constructed examples[17] of functions $\chi(z_1\cdots z_5;z_6)$, which have the properties described in the text and are such that for $z_1=z_2=m^2$, $z_3=z_4=\gamma$, $z_5=-4\Delta^2$, $z_6=4t^2$ they have singularities for $\Delta^2\geq G(t,\gamma;a,b)$, where $a\geq b$ and

$$G(t,\gamma;a,b)=\left[\left(t^2+\tfrac{1}{2}ab-\frac{m^2+\gamma}{4}\right)^2 \\ -\left(\frac{a+b}{2}t-\frac{a-b}{2}\frac{m^2-\gamma}{4t}\right)^2\right]\left[t^2-\left(\frac{a-b}{2}\right)^2\right]^{-1}.$$

In the equal mass case we have $a=b=2m$, $\gamma=m^2$, $t\geq m$ and find $G_{\min}=2m^2$ for $t=\sqrt{3}m/\sqrt{2}$. *Hence our limit* $\Delta_{\max}^2=2m^2$ *cannot be improved.* For pion-nucleon scattering the parameters are $a=m+\mu$, $b=3\mu$, $\gamma=\mu^2$, $t\geq\frac{1}{2}(m+\mu)$. The minimum of G is at $t=\frac{1}{2}(m+\mu)$ [μ/m=experimental mass ratio], and we obtain $G_{\min}(\pi)=\Delta_{\max}^2 =(8\mu^2/3)(2m+\mu/2m-\mu)$. In order to prove that the points $z_1\cdots z_5$ given above are contained in the region of holomorphy for all $t\geq\frac{1}{2}(m+\mu)$ and $2\mu^2<\Delta^2<G_{\min}(\pi)$, we go back to the vectors q_1, q_2 (see Sec. 3) and use a general representation for functions which are analytic in $W\cup N$, where N is some complex neighborhood of the real set $S=[q: (q_0+t)^2-\mathbf{q}^2<a^2, (q_0-t)^2-\mathbf{q}^2<b^2]$. [A proof for such representations has been given by F. J. Dyson, Phys. Rev. (to be published); see also reference 17 and L. Garding and A. Wightman (to be published).] We find that we have analyticity for $\Delta^2<G(t,\gamma;a,b)$, where G is the same quantity we obtain from our examples. Hence *the dispersion relations for pion-nucleon scattering can be proved for momentum transfers* 2Δ *in the center-of-mass system, which are smaller than* $4\mu(\frac{2}{3})^{\frac{1}{2}}[(2m+\mu)/(2m-\mu)]^{\frac{1}{2}}$. At present it is not known whether a discussion of the complete envelope of holomorphy of the four-body Green's function (a function of six *complex* variables) will lead to an improvement of our limit. Results corresponding to those described above have been obtained also by H. Lehmann (private communication).

[17] These examples were inspired by the paper of R. Jost and H. Lehmann, Nuovo cimento 10, 1598 (1957). We are indebted to Professor Lehmann for bringing this paper to our attention.

sponding $\chi(z_1,\cdots;z_6)$ we find, using Eq. (3.16),

$$\chi(z_1,\cdots,z_5;4t^2)=\frac{\rho(t)}{[2t^2-2(m+\mu)t+(m+\mu)^2-\frac{1}{2}(z_1+z_3)]^n[2t^2-2(m+\mu)t+(m+\mu)^2-\frac{1}{2}(z_2+z_4)]^n}. \tag{4.5}$$

These examples show that those points $z_1,\cdots,z_5$, which give rise to the limitation (4.3), lie on the envelope of holomorphy. Using only causality and spectral conditions we cannot hope to continue beyond these points. But there are certain symmetry properties of the four-body Green's function which we have not explored. We cannot exclude the possibility that these permit a further continuation in the relevant region.

The problem of proving a dispersion formula for the *meson-nucleon vertex function* $F[(k-p)^2,k^2,p^2]=\langle k|j(0)|p\rangle(4k_0p_0)^{\frac{1}{2}}$, is intimately related to the problem of deriving nucleon-nucleon dispersion relations. In essence we need only disregard the second four-vector variable q_2 in the discussions of Sec. 3. Then we can prove the representation $[k^2<(m+\mu)^2$, $p^2<(m+\mu)^2]$

$$F(z,k^2,p^2)=\frac{z^N}{\pi}\int_{(3\mu)^2}^{\infty}d\sigma^2\frac{\rho(\sigma^2,k^2,p^2)}{(\sigma^2-z)\sigma^{2N}}+\sum_{n=0}^{N-1}C_n(k^2,p^2)z^n \tag{4.6}$$

for $k^2+p^2<(m+\mu)^2$, and the requirement $k^2=p^2=m^2$ leads again to the condition $\mu>(\sqrt{2}-1)m$. In Eq. (4.5) we have omitted spin and isotopic spin variables, which are unimportant for the analytic properties of the vertex. The lower limit of the σ^2 integral corresponds to the case of pseudoscalar mesons; for scalar mesons it is $(2\mu)^2$. An example corresponding to Eq. (4.4) shows that we cannot improve this limit. However, in view of the symmetry conditions, an extension of the region of analyticity of $F(z_1,z_2,z_3)$ at the relevant points cannot be excluded. In any case, Jost[18] has given an example for $F(z_1,z_2,z_3)$, which satisfies spectral and causality conditions and is completely symmetric in all three variables. This example shows that, even including the symmetry conditions, one cannot derive the representation (4.6) for $F(z,m^2,m^2)$ if $\mu<(2/\sqrt{3}-1)m$. This value is above the experimental mass ratio. At present it is not clear, what further conditions one has to impose in order to assure the validity of the dispersion formula (4.6) for the pion-nucleon vertex. It has been shown by Nambu[19] that this relation holds in perturbation theory in every finite order.

Finally we would like to mention that our methods may also be applied to the case of *dispersion relations for K-meson-nucleon scattering*. Here we have again an unphysical continuum due to states of one Λ particle and one or more pions. The parameters of Theorem 1 are

$$a=M_K+2\mu,\quad b=m+\mu,\quad c=M_\Lambda+\mu,\quad \gamma_{\mathrm{phys}}=m^2,$$

and for $\gamma=m^2$ there is a small region near $t=\frac{1}{2}(M_\Lambda+\mu)$, for which the corresponding points (q_1,q_2) are not included in the envelope of holomorphy.

ACKNOWLEDGMENTS

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APPENDIX

In this Appendix we prove the "edge of the wedge" theorem, which has been extensively used in Sec. 3. It is convenient to prove first Lemma 1.

Lemma 1.—Let the function $f(z_0,z_1)$ of two complex variables (z_0,z_1) be given as the Fourier transform of two tempered distributions $\tilde{f}_r$ and $\tilde{f}_a$ such that it is analytic in the "wedge" W,

$$W=[(z_0,z_1):|y_1|<|y_0|,\ |x_0|<\infty,\ |x_1|<\infty].$$

For a given domain E in the (x_0,x_1) plane, we say that a sequence of pairs of complex numbers (z_{0n},z_{1n}) is an "E-limiting sequence" if it satisfies the following conditions,

(1) $\lim_n y_{0n}=\lim_n y_{1n}=0$,

(2) $\lim_n(x_{0n},x_{1n})\epsilon E$,

(3) there is a number $c>1$, and independent of n, so that for all n $|y_{0n}|>c|y_{1n}|$. Then we assume that $f(z_0,z_1)$ has the limiting property that for any E-limiting sequence the limit $\lim_n f(z_{0n},z_{1n})$ exists and is independent of the particular sequence, only depending on the limit point.

We wish to prove that $f(z_0,z_1)$ is also analytic in some neighborhood N of the set

$$S=[(z_0,z_1):y_0=y_1=0,\ (x_0,x_1)\epsilon E].$$

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[18] R. Jost (private communication). Professor Jost was so kind to permit us to quote his example. We write it in the form

$$F_J(z_1,z_2,z_3)=f(z_1,z_2,z_3)+f^*(z_1^*,z_2^*,z_3^*),$$

where

$$f(z_1,z_2,z_3)=[(1+a^2-z_1m^{-2})^{\frac{1}{2}}+(1+a^2-z_2m^{-2})^{\frac{1}{2}}+(1+a^2-z_3m^{-2})^{\frac{1}{2}}-b-ic]^{-1},\quad (a>0,b>0,c>0).$$

The conditions on the parameters are

$$b>2a,\quad 1+a^2-(b+c)^2>0.$$

Since we may take c as small as we please, these two inequalities imply $a^2<\frac{1}{3}$. Taking $m(1+a^2)^{\frac{1}{2}}=m+\mu$, we have to require $\mu>(2/\sqrt{3}-1)m$ in order to avoid a contradiction with a dispersion relation of the form (4.6) for $z_2=z_3=m^2$. The condition $1+a^2-(b+c)^2>0$ is sufficient to assure that F_J is analytic in the domain obtained on the basis of causality, spectrum and symmetry. For the proof the region of analyticity obtained by G. Källén and A. S. Wightman is very useful [*Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1957)].

[19] Y. Nambu, Nuovo cimento (to be published).

To achieve this, we shall prove analyticity in a neighborhood of each point of S. Since the origin of the real coordinates (x_0,x_1) has not been fixed, we may take this origin to be at the particular point of E that we are considering. Thus we wish to prove analyticity at $z_0=z_1=0$, where $(0,0)\epsilon E$.

If we suppose that $f(z_0,z_1)$ is analytic in some neighborhood of $z_0=z_1=0$, then for some r and $|z_0|<r$, $|z_1|<r$, $f(z_0,z_1)$ has the uniformly convergent power series expansion

$$f(z_0,z_1)=\sum_{m,n=0}^{\infty} a_{mn}z_0{}^m z_1{}^n. \qquad \text{(A1)}$$

On the analytic plane Π_α: $z_0=\alpha_0\lambda$, $z_1=\alpha_1\lambda$, with α_0, α_1 real and $|\alpha_1/\alpha_0|<1$, then

$$f(z_0,z_1)=\sum_{m,n} a_{mn}\alpha_0{}^m\alpha_1{}^n\lambda^{m+n}$$

$$=\sum_{\rho=0}^{\infty}\lambda^\rho\sum_{n=0}^{\rho} a_{\rho-n,\,n}\alpha_0{}^{\rho-n}\alpha_1{}^n=\sum_{\rho=0}^{\infty}c_\rho\lambda^\rho, \qquad \text{(A2)}$$

where the rearrangement from the second to the third expression in Eq. (A2) is permitted since the power series is absolutely and uniformly convergent. The coefficients c_ρ are obtained in terms of the coefficients a_{mn} as

$$c_\rho=\sum_{n=0}^{\rho} a_{\rho-n,\,n}\alpha_0{}^{\rho-n}\alpha_1{}^n. \qquad \text{(A3)}$$

The method we shall use here is to reverse this procedure and determine the a_{nm} from the c_ρ for different Π_α. We may take $\alpha_0=1$ without loss of generality, and write α in place of α_1. Any analytic plane Π_α with real α satisfying $|\alpha|<1$ lies completely inside W, except for $\text{Im}\lambda=0$. Hence, by our initial assumption, $f(z_0,z_1)$ is analytic in Π_α except possibly on the real λ axis. The limiting property assumed for $f(z_0,z_1)$ is equivalent to requiring that $f(z_0,z_1)$ has the same limit as we approach the real axis in Π_α either from above or below, and similarly for the partial derivatives of $f(z_0,z_1)$, provided $(\lambda,\alpha\lambda)\epsilon E$. Since $(0,0)\epsilon E$, $|\alpha|<1$, and E is open, then $(\lambda,\alpha\lambda)\epsilon E$ for a small open interval of λ near $\lambda=0$. Thus $f(z_0,z_1)$ is analytic in a small neighborhood of the origin in Π_α. Then there is some small positive number d so that we may expand $f(z_0,z_1)$ as in Eq. (A2) for $(z_0,z_1)\epsilon\Pi_\alpha$ and $|\lambda|<d$. We note that d may be chosen to be independent of α for $|\alpha|<1$.

It is immediate that $f(z_0,z_1)$ is bounded on any bounded subset of W which has a positive distance from the boundary of W. In addition, one can show by distribution methods using the Fourier representation that there is a positive number M, depending only on c and d, such that

$$|f(z_0,z_1)|<M(c,d)$$

for $(z_0,z_1)\epsilon\Pi_\alpha$, $|\alpha|<c<1$, $|\lambda|<d$. Then by Cauchy's inequality

$$|c_\rho|<M(c,d)/d^\rho. \qquad \text{(A4)}$$

For $|\alpha|<c$, the coefficients $c_\rho(\alpha)$ are polynomials in α of order ρ,

$$c_\rho(\alpha)=\sum_{n=0}^{\rho} a_{\rho-n,\,n}\alpha^n. \qquad \text{(A5)}$$

In order to verify this statement, we have to show that $d^{\rho+1}c_\rho(\alpha)/d\alpha^{\rho+1}=0$ for $\rho=1, 2, \cdots$. The equation for $\rho=0$ is evident, since

$$c_0(\alpha)=\lim_{\lambda\to 0}[f(z_0,z_1)]_{z_0=\lambda,\ z_1=\alpha\lambda}$$

is independent of α. For $\rho=1$, we have then

$$\frac{dc_1(\alpha)}{d\alpha}=\lim_{\lambda\to 0}\left[\frac{\partial f(z_0,z_1)}{\partial z_1}\right]_{z_0=\lambda,\ z_1=\alpha\lambda}, \qquad \text{(A6)}$$

which again is independent of α. This follows from the Fourier representation and the E-limiting property of $f(z_0,z_1)$. The proof for higher ρ proceeds by induction on ρ.

Equation (A5) may now be solved for the a_{mn} in terms of the $c_\rho(\alpha)$ and α. This requires the use of a number of different analytic planes. The first few equations of (A5) are

$$c_0=a_{00},\quad c_1=a_{10}+a_{01},\quad c_2=a_{20}+a_{11}\alpha+a_{02}\alpha^2.$$

To determine a_{10} and a_{01}, we also have

$$c_1(\alpha')=a_{10}+a_{01}\alpha',$$

and so

$$a_{01}=[c_1(\alpha')-c_1(\alpha)]/(\alpha'-\alpha)$$

$$a_{10}=[\alpha'c_1(\alpha)-\alpha c_1(\alpha')]/(\alpha'-\alpha),$$

provided $\alpha'\neq\alpha$. Evidently such α' and α can be chosen to satisfy $\alpha'\neq\alpha$, $|\alpha'|<c$ and $|\alpha|<c$. In general we may determine the quantities a_{n0}, $a_{n-1,1}$, $\cdots$, $a_{n-r,r}$, $\cdots$, $a_{0,n}$ in terms of $c_n(\alpha)$ for n analytic planes $\Pi_{\alpha 1}$, $\cdots$, $\Pi_{\alpha n}$, provided that the corresponding determinant.

$$\prod_{n\geq i\geq j\geq 1}(\alpha_i-\alpha_j)\neq 0.$$

Such $\alpha_1\cdots\alpha_n$ can evidently be chosen in the interval $(-c,+c)$ for any n.

Equation (A4), written in terms of the a_{mn}, now becomes

$$\left|\sum_{n=0}^{\rho} a_{\rho-n,\,n}\alpha^n\right|<\frac{M(c,d)}{d^\rho} \qquad \text{(A7)}$$

for $\rho=0, 1, 2, \cdots$. We wish to obtain bounds on the coefficients a_{mn} so that on replacing the c_ρ in Eq. (A2) by their expression in terms of the a_{mn} as given in Eq. (A5) the resulting series of Eq. (A2) may be shown to be absolutely convergent for small enough $|z_0|$, $|z_1|$, and so rearranged to give the expansion of Eq. (A1).

Let us consider the polynomial

$$P(x)=\sum_{n=0}^{N} b_n x^n,$$

and we assume that for real x satisfying $|x|\leq c$, then

$$|P(x)|\leq M.$$

We wish to determine bounds on the coefficients b_n. The Cauchy inequalities cannot be used here, since though we may immediately extend $P(x)$ to $P(z)$ for complex z the bound on $P(x)$ may not extend to the inside of the circle $|z|=c$.

We define[20]

$$f(z)=P(z)[z+i(c^2-z^2)^{\frac{1}{2}}]^{-N}.$$

Then $f(z)$ is a bounded analytic function in the whole complex z plane, including infinity, with a cut from $-c$ to $+c$ along the real axis. By the maximum modulus theorem,

$$|f(z)|\leq \operatorname*{Max}_{|x|\leq c}|f(x)|\leq M/c^N.$$

The Cauchy inequality for b_n is thus

$$|b_n|<\frac{1}{2\pi}\left|\int_{|z|=R}\frac{f(z)}{z^{n+1}}[z+i(c^2-z^2)^{\frac{1}{2}}]^N dz\right|$$

$$\leq\frac{M}{R^n c^N}\sup_{|z|=R}|z+i(c^2-z^2)^{\frac{1}{2}}|^N\leq\frac{M}{c^N R^n}[R+(R^2+c^2)^{\frac{1}{2}}]^N.$$

Taking $R=c$, we obtain

$$|b_n|\leq M3^N/c^n;$$

hence we have

$$|a_{\rho-n,\,n}|<M(c,d)3^\rho/d^\rho c^n.$$

Then

$$\left|\sum_{\rho=0}^{\infty} z_0{}^\rho \sum_{n=0}^{\rho} a_{\rho-n,\,n} z_1{}^n z_0{}^{-n}\right|$$

$$\leq\sum_{\rho=0}^{\infty}|z_0|^\rho M(c,d)\frac{3^\rho}{d^\rho}\sum_{n=0}^{\rho}\left|\frac{z_1}{cz_0}\right|^n. \quad \text{(A8)}$$

If $|z_0|<\frac{1}{3}d$, $|z_1|<\frac{1}{3}cd$, the right-hand side of Eq. (A8) is less than

$$M(c,d)\sum_{\rho=0}^{\infty}\rho d^\rho,$$

and so is finite provided $d<1$ (which is assumed). Hence the series $(\sum_{\rho=0}^{\infty} z_0{}^\rho \sum_{n=0}^{\rho} a_{\rho-n,\,n} z_1{}^n z_0{}^{\rho-n})$ is absolutely and uniformly convergent in $|z_0|<\frac{1}{3}d$, $|z_1|<\frac{1}{3}dc$, and so may be rearranged inside this region to give $\sum_{m,\,n=0}^{\infty} a_{mn} z_0{}^m z_1{}^n$.

So it has been shown that the power series expansions on the analytic planes Π_α can be joined together to give a power series expansion, convergent for $|z_0|<\frac{1}{3}d$, $|z_1|<\frac{1}{3}cd$, which equals these separate expansions on the various planes, and so equals $f(z_0,z_1)$ from which we started.

Hence we have proved analyticity of $f(z_0,z_1)$ in a small neighborhood of $z_0=z_1=0$.

We now consider a function f of four complex variables $z=(z_0,\mathbf{z})=(z_0,z_1,z_2,z_3)$ and prove Lemma 2.

Lemma 2.—Let the function $f(z)$ be analytic in the four complex variables z_0, $\mathbf{z}$ in the wedge W in four dimensions,

$$W=[(z_0,\mathbf{z}):|y_0|>|\mathbf{y}|,\ |x_0|<\infty,\ |\mathbf{x}|<\infty],$$

where it is the Fourier transform of two tempered distributions. We assume that $f(z)$ has the limiting property of Lemma 1, extended by replacing z_1 by $\mathbf{z}$. Then we can find a neighborhood N of the set

$$S=[z:y_0=0,\ \mathbf{y}=0,\ x\epsilon E],$$

so that $f(z)$ can be continued analytically throughout $W\cup N$.

The proof of this lemma is obtained by simple extension of the proof of Lemma 1.

We now prove the "edge of the wedge" theorem.

"Edge of the Wedge" Theorem.—Let $f(z,z')$ be a function of 8 complex variables,

$$z=(z_0,z_1,z_2,z_3)=(z_0,\mathbf{z}),\quad z'=(z_0',\mathbf{z}').$$

We suppose that $f(z,z')$ is analytic in the double wedge $W\times W$,

$$W=[z:|y_0|>|\mathbf{y}|,\ |x_0|<\infty,\ |\mathbf{x}|<\infty],$$

where it is the Fourier transform[21] of tempered distributions. We further assume that $f(z,z')$ has the limiting property that for any E-limiting sequence of complex numbers $z_n=(z_{0n},\mathbf{z}_n)$, then

$$\lim_n[f(z_n,z')]|_{z'\epsilon W\cup R_4}\quad\text{and}\quad\lim_n[f(z,z'_n)]|_{z\epsilon W\cup R_4}$$

exist, and are independent of the particular sequence, only depending on the limit point.

We wish to prove that there is some neighborhood N of the set

$$S=[z:y=0,\ x\epsilon E],$$

so that $f(z,z')$ may be analytically continued to $(W\cup N)\times(W\cup N)$.

Proof.—We may prove the theorem by applying the method used in the proof of Lemma 1 to the variables z and z' simultaneously. In order to avoid repetition, we give only a brief sketch of the generalization. We may consider z and z' as two-vectors, since the extension to four-vectors is straightforward.

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[20] Compare M. Riesz, Acta Math. 40, 43 (1916). During a series of lectures which one of us (R. O.) gave at the University of Maryland in July 1957, Professor Marcel Riesz kindly pointed out that he had done similar calculations in 1916.

[21] As an equivalent assumption we could require that $f(z,z')$ is bounded by a polynominal for $(z,z')\epsilon W\times W$, with (x,x'), restricted to any compact $K\epsilon R_8$. Compare L. Schwartz, Meddelanden fran Lunds Universitets, Supplementband, p. 196 (1952).

Consider a point in S, say (0,0), and the analytic planes $\Pi_{\alpha\beta}$ which are given by $z_0=\lambda$, $z_1=\alpha\lambda$; $z_0'=\lambda'$, $z_1'=\beta\lambda'$ with α, β real and $|\alpha|, |\beta|<c<1$. As in Lemma 1, we have

$$f(z,z')=\sum_{\rho,\sigma=0}^{\infty} c_{\rho\sigma}(\alpha,\beta)\lambda^\rho\lambda'^\sigma,\ (z,z')\epsilon\Pi_{\alpha\beta},\ |\lambda|,|\lambda'|<d;$$

and

$$c_{\rho\sigma}(\alpha,\beta)=\sum_{n=0}^{\rho}\sum_{m=0}^{\sigma} A_{\rho-n,\,n;\,\sigma-m,\,m}\alpha^n\beta^m. \qquad \text{(A9)}$$

Equation (A8) follows from the E-limiting properties of $f(z,z')$; it can be solved for the $A_{r,\,n;\,s,\,m}$ by using a sequence of different planes $\Pi_{\alpha\beta}$. As in Lemma 1, we obtain $|f(z,z')|<M(c,d)$ for $(z,z')\epsilon\Pi_{\alpha\beta}$; $|\lambda|$, $|\lambda'|<d$ and find $|A_{\rho-n,\,n;\,\sigma-m,\,m}|<M(c,d)3^{\rho+\sigma}/d^{\rho+\sigma}c^{n+m}$. Consider now the series $\sum_{rn,\,sm}A_{rn,\,sm}z_0{}^r z_1{}^n z_0{}'^s z_1{}'^m$; it is convergent for $|z_0|$, $|z_0'|<\frac{1}{3}d$, $|z_1|$, $|z_1'|<\frac{1}{3}cd$ and equal to $f(z,z')$ on an infinite sequence of planes $\Pi_{\alpha\beta}$. Hence the series represents a continuation of $f(z,z')$. Performing the corresponding construction for all $(z,z')\epsilon S$ we obtain a continuation to $(W\cup N)\times(W\cup N)$.

In the "edge of the wedge theorem" we proved that we have analyticity of $f(z,z')$ at each point of $(W\cup N)\times(W\cup N)$, for some neighborhood N of the set S. The neighborhood N is the union of all $N(X)$ with $X\epsilon E$, where $N(X)$ is some neighborhood of the point $z=X$. The dependence of N upon X is as yet arbitrary, except that $N(X)$ vanishes as X tends to the boundary of E. In Sec. 3 we find it useful to take for E the set $E=[x: |x_0|<\eta(t), |\mathbf{x}|<\infty]$.

We must construct a semitube from $W\cup N$ in order to apply the semitube method. Since W is already a tube, then it is necessary to construct a semitube H contained in N. Because $\mathbf{x}$ is arbitrary while x_0 is restricted, in E, then we expect H to be a semitube of the form

$$H=[z: |\mathbf{x}|<\infty,\ |\mathbf{y}|<v(z_0),\ |x_0|<\eta(t),\ |y_0|<\epsilon]$$

for some small ϵ. The quantity $v(z_0)$ will tend to zero as $|x_0|$ tends to $\eta(t)$. We have $H\subset N$ only if $N(X)$ is independent of $\mathbf{X}$. This independence does not occur for *any* function analytic in S, as is easy to show by simple examples. We cannot conclude that there exists a semitube $H\subset N$ without using analyticity in W.

We wish to show that f is analytic in some semitube neighborhood H of the set S. It is evident from what we have said above that, to do this, we will have to continue f from $W\cup N$ to $W\cup H$. We achieve this continuation by the direct method developed by one of us (H. J. B.).[7]

For simplicity we replace the three-vector by the complex number z_1. By a simple extension our result in terms of z_1 may be immediately generalized to the case $\mathbf{z}$ instead of z_1.

We take the analytic plane $z_1=X_1$. In the z_0 plane the domain of analyticity of our function is the cut plane, with cuts running from $\eta(t)$ to $+\infty$, and $-\eta(t)$ to $-\infty$. In the cut plane we take any point x_0, with $|x_0|<\eta(t)$. Furthermore we take the circle $c(x_0)$ in that plane, with center x_0 and radius r less than $\eta(t)-|x_0|$. We may assume $N(X_1)$ to be the region

$$N(X_1)=[(z_0,z_1): x_1=X_1,\ |y_0|<\epsilon,\ |y_1|<\epsilon,\ |x_0|<\eta(t)-\delta(\epsilon,X_1)],$$

where $\delta(\epsilon,X_1)\rightarrow 0$ as $\epsilon\rightarrow 0$. Then we have that N is the union of the $N(X_1)$ for all X_1. We define $D=W\cup N$, with $N=\cup N(X_1)$, where $N(X_1)$ is given above. The circle $c(x_0)$ is completely in D. The Euclidean distance $\delta_D(z_0)$ of any point z on $c(x_0)$ from the boundary of D is then $\delta_D(z_0)=|y_0|/\sqrt{2}+O(\epsilon)$. We evaluate $h(\epsilon)=(1/2\pi)\int_0^{2\pi}\log\delta_D(re^{i\theta})d\theta$ and find $h(\epsilon)=(r/2\sqrt{2})+O(\epsilon)$. Now the theorem of reference 7 allows us to analytically continue $f(z_0,z_1)$ into the sphere $\|z-Z\|<h(\epsilon)$ about the point Z with $Z_0=0$, $Z_1=X_1$. Since ϵ is arbitrarily small and r is as close to $[\eta(t)-|x_0|]$ as we please, we may continue to the sphere $\|z-Z\|<(1/2\sqrt{2})[\eta(t)-|x_0|]$. In particular, we have analyticity in the set

$$H(X_1)=[z: x_1=X_1,\ |y_1|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |y_0|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |x_0|<\eta(t)],$$

and hence

$$H=\cup H(X_1)=[z: |x_1|<\infty,\ |y_1|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |y_0|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |x_0|<\eta(t)].$$

H is now a semitube.

Lemma 3.—Let $f(z,z')$ be a function satisfying the conditions of the "edge of the wedge" theorem. Then we can find a neighborhood H of the set $S=[z: y_\mu=0, |x_0|<\eta(t), |\mathbf{x}|<\infty]$ which is of the form

$$H=[z: |\mathbf{x}|<\infty,\ |\mathbf{y}|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |y_0|<\tfrac{1}{4}(\eta(t)-|x_0|),\ |x_0|<\eta(t)],$$

and such that $f(z,z')$ may be analytically continued into $(W\cup H)\times(W\cup H)$.

To prove this lemma, we use the above process to continue $f(z,z')$, as a function of z, to $W\cup H$. We can show that $f(z,z')$ is an analytic function of (z,z') together in $(W\cup H)\times(W\cup S)$. We continue on z' now to $W\cup H$, and again we have that $f(z,z')$ is an analytic function of (z,z') together in $(W\cup H)\times(W\cup H)$. This proves the lemma.

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INTEGRAL REPRESENTATION OF CAUSAL COMMUTATORS

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Summary. The structure of the commutator of field operators will be investigated. An integral representation will be given for the matrix element between eigenstates of the momentum operator. For this we consider microscopic causality and the spectrum of the momentum operator.

1. PROPERTIES OF A CAUSAL COMMUTATOR

As is known, causality in field theory is expressed by the vanishing of the commutator (or anticommutator) of appropriately chosen field quantities for space-like intervals. In the following, we want to treat some mathematical methods which might be of interest in the investigation of the formulation of the causality condition.

For the sake of simplicity, we shall discuss scalar field quantities A(x), B(x). Then we have:

$$[A(x), B(x')] = 0 \quad \text{falls} \quad (x - x')^2 = (\boldsymbol{x} - \boldsymbol{x}')^2 - (x_0 - x_0')^2 > 0 \,. \tag{1.1}$$

From this the matrix element

$$\langle P, \alpha | [A(x), B(x')] | Q, \beta \rangle = 0 \,, \qquad \text{for } (x - x')^2 > 0. \tag{1.2}$$

also vanishes. The states $|P, \alpha\rangle$ and $|Q, \beta\rangle$ should be, in this case, eigenstates of the energy-momentum vector for the eigenstates P and Q. α and β are additive constants which uniquely specify the states.

Obviously we have $P_0 \geq 0$, $P^2 \leq 0$ and $Q_0 \geq 0$, $Q^2 \leq 0$; that is, P and Q lie in the future light-cone.

By using translational invariance, we obtain for the matrix element (1.2)

$$\langle P, \alpha | [A(x), B(x')] | Q, \beta \rangle = \\ = \exp\left[i(Q - P)\frac{x + x'}{2}\right] \langle P, \alpha | [A(\xi/2), B(-\xi/2)] | Q, \beta \rangle \,. \tag{1.3}$$

Here we set $\xi = x - x'$.

According to this, it is enough to discuss the function

$$F(\xi) = i\langle P, \alpha|[A(\xi/2), B(-\xi/2)]|Q, \beta\rangle \tag{1.4}$$

In addition to causality, the structure of the eigenvalue spectrum of the energy-momentum vector gives an important condition for $F(\xi)$. By summing over intermediate states, we obtain

$$\begin{aligned} F(\xi) &= \frac{1}{(2\pi)^4}\int d^4k \left\{\exp\left[i\xi\left(k - \frac{P+Q}{2}\right)\right]\tilde{G}_1(k) - \right. \\ &\qquad \left. - \exp\left[-i\xi\left(k - \frac{P+Q}{2}\right)\right]\tilde{G}_2(k)\right\} = \\ &= \frac{1}{(2\pi)^4}\int d^4q \exp[iq\xi]\left\{\tilde{G}_1\left(q + \frac{P+Q}{2}\right) - \tilde{G}_2\left(-q + \frac{P+Q}{2}\right)\right\} . \end{aligned} \tag{1.5}$$

Here the functions $\tilde{G}_1(k)$ and $\tilde{G}_2(k)$ are different from zero only when $\underline{k}$ belongs to the spectrum of the energy-momentum vector; that is, only when $\underline{k}$ lies in the future light-cone.

The function $F(\xi)$ has thus the following significant properties:

a) $F(\xi)$ disappears outside the light-cone.

b) The Fourier-transform $F(q) = \int d^4\xi \exp[-iq\xi]F(\xi)$ vanishes outside the region in the future light-cone with the vertex at $-\left(\frac{P+Q}{2}\right)$ and in the past light-cone with the vertex at $\left(\frac{P+Q}{2}\right)$.[1]

The attempt to obtain, as general as possible, a class of functions with the properties (a) and (b) by means of a representation, formed the motive for the present investigation. However, in the following, we will generalize the statement of the problem. The corresponding results seem to be useful.

We also notice that obviously no essential restriction is meant by setting $\left(\frac{P+Q}{2}\right) = (1, 0, 0, 0)$. The case $(P + Q)^2 = 0$ which is thus excluded will not be of great significance.

2. THE MATHEMATICAL STATEMENT OF THE PROBLEM

We consider the following as a useful generalization and definition of the question raised in Section 1.

Let $F(\xi) = F_R(\xi) - F_A(\xi)$ where $F_R(\xi)$ and $F_A(\xi)$ are distributions[2] which vanish outside the future and past light-cones respectively. Furthermore, it should be known that $\tilde{F}(q) = \int d^4\xi \exp[-iq\xi]F(\xi)$ should vanish in the double cone $|q_0| + |\mathbf{q}| < 1$. We are looking for a representation for $\tilde{F}(q)$.

The connection between the old and the new representation is that the complement of the region of the future light-cone starting from $(-1, 0, 0, 0)$ and of the past light-cone starting from $(1, 0, 0, 0)$, that is, the region in which $\tilde{F}(q)$, in Section 1, vanishes, can be represented as the combination of the two cones $|q_0| + |\mathbf{q} - \mathbf{a}| < |\mathbf{a}| - 1$.

It should of course be stated here that we shall not treat the above-formulated mathematical problem rigorously. Our investigation makes modest mathematical claims, but should be sufficient for the purposes of physics.

From the fact that the double cone $|q_0| + |\mathbf{q}| < 1$ is symmetric to the plane $q_0 = 0$, it follows that we can decompose $F(\xi_0,\boldsymbol{\xi})$ into a symmetric part $\frac{1}{2}[F(\xi_0,\boldsymbol{\xi}) + F(-\xi_0,\boldsymbol{\xi})]$ and an antisymmetric part $\frac{1}{2}[F(\xi_0,\boldsymbol{\xi}) - F(-\xi_0,\boldsymbol{\xi})]$ both of which satisfy the original assumptions (including the divisibility into retarded advanced distributions).

Further, it becomes clear that the symmetric part can be written as a derivative of ξ_0 of an antisymmetric distribution which again satisfies all the assumptions. (See Appendix.)

Thus, it is enough to give a representation for the antisymmetric $F(\xi)$. This we can give without restriction as follows:

$$F(\xi_0, \boldsymbol{\xi}) = F_R(\xi_0, \boldsymbol{\xi}) - F_R(-\xi_0, \boldsymbol{\xi}) \,. \tag{2.1}$$

Now we proceed from the assertion that an invariant in ξ relative to the orthochronous Lorentz group and an antisymmetric distribution in ξ_0 vanishes outside the light-cone. The same is true for its Fourier-transform. Thus $L(\xi;\eta)$ should be such a distribution in ξ, and in η, $L(\xi;\eta)$ should be infinitely differentiable and every derivative should be bounded by a suitable polynomial in $\boldsymbol{\eta}$.[3] Now

$$F(\xi_0, \boldsymbol{\xi}) = L(\xi_0, \boldsymbol{\xi}; \boldsymbol{\xi}) \tag{2.2}$$

should be a fully sufficient class of antisymmetric distributions which vanish outside the light-cone.

By means of a Fourier-transformation, we obtain from (2)

$$\widetilde{F}(q_0, \boldsymbol{q}) = \int \mathrm{d}^3 u \, \widetilde{L}(q_0, \boldsymbol{q} - \boldsymbol{u}; \boldsymbol{u}), \tag{2.3}$$

where

$$\widetilde{L}(q; \boldsymbol{u}) = (2\pi)^{-3} \int \mathrm{d}^4\xi \int \mathrm{d}^3\eta \exp[-iq\xi - i\boldsymbol{u}\boldsymbol{\eta}] \, L(\xi; \boldsymbol{\eta}) \tag{2.4}$$

is odd in q_0, an orthochronous Lorentz invariant distribution in q, and a strongly decreasing distribution in $\mathbf{u}$.[4]

Equations (2.2) and (2.3) are completely equivalent, so that we can understand (2.3) as the condition for the vanishing of $F(\xi_0,\boldsymbol{\xi})$ outside the light-cone. It is now a matter of choosing $\widetilde{L}(q;\mathbf{u})$ in such a way that $\widetilde{F}(q_0,\mathbf{q})$ vanishes in the double cone $|q_0| + |\mathbf{q}| < 1$.

3. THE VANISHING OF $\widetilde{F}(q_0,\mathbf{q})$ IN THE DOUBLE CONE

We now let

$$\widetilde{L}(q; \boldsymbol{u}) = \varepsilon(q_0) \, \hat{\Phi}(q_0^2 - \boldsymbol{q}^2, \boldsymbol{u}) \tag{3.1}$$

where $\hat{\Phi}$ vanishes when the first argument is negative.[5]

$\hat{\Phi}(\kappa^2,\mathbf{u})$ should be a distribution in κ^2. We restrict ourselves to distributions L which can be represented as in (3.1).

Our goal is to prove the following theorem:

Theorem 1: Necessary and sufficient conditions for $F(q_0,\mathbf{q})$ (2.3) in $|q_0| + |\mathbf{q}| < 1$ to vanish are that $\hat{\Phi}(\kappa^2,\mathbf{u})$ in $\kappa^2 + \mathbf{u}^2 < 1$ also should vanish.

Proof:

a) The condition is sufficient: if we substitute (3.1) in (2.3), we obtain

$$\tilde{F}(q_0, \boldsymbol{q}) = \varepsilon(q_0)\int \mathrm{d}^3u\, \hat{\Phi}(q_0^2 - (\boldsymbol{q} - \boldsymbol{u})^2, \boldsymbol{u})\,. \tag{3.2}$$

Thus, $\hat{\Phi}(\kappa^2,\mathbf{u})$ will be integrated over the cone $\kappa^2 + (\mathbf{u} - \mathbf{q})^2 = q_0^2$. This cone is completely contained in the cone $\kappa^2 + \mathbf{u}^2 \leq (|q_0| + |\mathbf{q}|)^2$. The assertion follows from this.

b) The condition is necessary: we show this first for continuous $\hat{\Phi}(\kappa^2,\mathbf{u})$. Thus, we have to show that from the vanishing of

$$\tilde{F}_1(q_0^2, \boldsymbol{q}) = \int \mathrm{d}^3u \int \mathrm{d}\varkappa^2\, \delta(q_0^2 - \varkappa^2 - (\boldsymbol{q} - \boldsymbol{u})^2)\, \hat{\Phi}(\varkappa^2, \boldsymbol{u}) \tag{3.3}$$

in $|q_0| + |\mathbf{q}| < 1$, it follows that $\hat{\Phi}(\kappa^2,\mathbf{u}) = 0$ for $\kappa^2 + \mathbf{u}^2 < 1$ and indeed for continuous $\hat{\Phi}$.

It seems more convenient to perform a transformation of variables:

$$\left\{\begin{aligned} \eta_0 &= \frac{1 - \varkappa^2 - \boldsymbol{u}^2}{1 + \varkappa^2 + \boldsymbol{u}^2}\,; \quad \varkappa^2 = \frac{1 - \eta_0^2 - \boldsymbol{\eta}^2}{(1 + \eta_0)^2}\,; \\ \boldsymbol{\eta} &= \frac{2\boldsymbol{u}}{1 + \varkappa^2 + \boldsymbol{u}^2}\,; \quad \boldsymbol{u}^2 = \frac{\boldsymbol{\eta}}{1 + \eta_0}\,. \end{aligned}\right. \tag{3.4}$$

Obviously we have

$$\eta_0^2 + \boldsymbol{\eta}^2 = \frac{(1 - \varkappa^2 - \boldsymbol{u}^2)^2 + 4\boldsymbol{u}^2}{(1 + \varkappa^2 + \boldsymbol{u}^2)^2} < 1\,. \tag{3.5}$$

Thus we have to concern ourselves in η-space only with the interior of the unit cone.

Further it results from the vanishing of $\tilde{F}_1$ in the double cone

$$\int d^4\eta\, \Psi(\eta)\, \delta(\gamma_0\eta_0 + \boldsymbol{\gamma}\boldsymbol{\eta} - \varrho) = 0\,, \tag{3.6}$$

where we set

$$\left\{\begin{aligned} \gamma_0 &= \frac{q_0^2 - \boldsymbol{q}^2 + 1}{\sqrt{(q_0^2 - \boldsymbol{q}^2 + 1)^2 + 4\boldsymbol{q}^2}}\,, \qquad \boldsymbol{\gamma} = \frac{2\boldsymbol{q}}{\sqrt{(q_0^2 - \boldsymbol{q}^2 + 1)^2 + 4\boldsymbol{q}^2}}\,, \\ \varrho &= \frac{1 + \boldsymbol{q}^2 - q_0^2}{\sqrt{(q_0^2 - \boldsymbol{q}^2 + 1)^2 + 4\boldsymbol{q}^2}} \end{aligned}\right. \tag{3.6'}$$

and

$$\Psi(\eta) = (1+\eta_0)^{-4} \hat{\Phi}(\varkappa^2, \boldsymbol{u}), \tag{3.7}$$

$\Psi(\eta)$ is to be set to zero for $\eta_0^2 + \eta^2 > 1$. Naturally (3.6) is valid only in the form of the double cone resulting from (3.6′), that is, for

$$\gamma_0^2 + \gamma^2 = 1\,, \quad \gamma_0 > 0\,, \quad \frac{\varrho}{\sqrt{1-\gamma_0^2}} > 1\,. \tag{3.8}$$

Clearly (3.6) and (3.8) means that the integral of $\Psi(\eta)$ over every plane which does not meet the cylinder $-1 \leq \eta_0 \leq 0$, $\eta^2 \leq 1$ vanishes. It then remains to be shown that from this follows the vanishing of $\Psi(\eta)$ for $\eta_0 > 0$. This, according to (3.7) and (3.4), is equivalent to the statement of Theorem 1.

This proof is based on the following auxiliary theorem.

Auxiliary theorem: The continuous function $\chi(\eta)$ should vanish outside a certain region. It should also have the property that its integral over every plane which does not meet a given finite (closed) convex region $\mathfrak{B}$ should vanish. Then $\chi(\eta)$ vanishes outside $\mathfrak{B}$.

In our case the region $\mathfrak{B}$ is the previously mentioned cylinder.

We postpone the proof of the auxiliary theorem. The general case of the theorem is related to the one we just treated, inasmuch as we substitute the distributions $F_1(q_0^2,q)$ and $\Phi(\kappa^2,u)$ with smooth functions. There should, perhaps, be an $\alpha(\kappa^2)$, an infinitely differentiable continuous function, which strongly decreases just as $\beta(u)$. Then let us define

$$\bar{\tilde{F}}_1(q_0^2, \boldsymbol{q}) = \int \mathrm{d}^3u \int \mathrm{d}\varkappa^2\, F_1(q_0^2 - \varkappa^2, (\boldsymbol{q} - \boldsymbol{u}))\, \alpha(\varkappa^2) \beta(\boldsymbol{u})\,. \tag{3.9}$$

Clearly we have

$$\bar{\tilde{F}}_1\, (q_0^2,, \boldsymbol{q}) = \int \mathrm{d}^3u\, \bar{\hat{\Phi}}(q_0^2 - (\boldsymbol{q} - \boldsymbol{u})^2, \boldsymbol{u}) \tag{3.10}$$

with

$$\Phi(\varkappa^2, \boldsymbol{u}) = \int \mathrm{d}^3v \int \mathrm{d}\lambda^2\, \hat{\Phi}(\varkappa^2 - \lambda^2, \boldsymbol{u} - \boldsymbol{v})\, \alpha(\lambda^2) \beta(\boldsymbol{v})\,. \tag{3.11}$$

We can now easily find a sequence α_k, β_k such that

a) $\bar{\tilde{F}}_{1k} \rightarrow \bar{\tilde{F}}_1$, $\bar{\hat{\Phi}}_k \rightarrow \bar{\hat{\Phi}}$ for $k \rightarrow \infty$.

b) $\bar{\tilde{F}}_{1k}$ vanishes in "shrunken" double cone $|q_0| + |q| < 1 - \varepsilon$.

On the basis of what was already proven, $\bar{\hat{\Phi}}_k$ vanishes in $\kappa^2 + u^2 - (1-\varepsilon)^2$. Since $\varepsilon > 0$ is here arbitrary $\hat{\Phi}$ thus vanishes in $\kappa^2 + u^2 < 1$.

There remains still the:

Proof of the Auxiliary Theorem:[6] For the sake of clarity, we refer next to the 2-dimensional case and we form a coordinate system as indicated in Fig. 1.

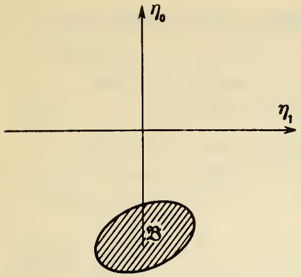

Fig. 1. Choice of the coordinate system in the proof of the Auxiliary Theorem.

The essential thing here is that the half-plane $\eta_0 \geq 0$ has a non-vanishing interval from $\mathfrak{B}$. Clearly we have

$$\int_0^\infty d\eta_0 \int d\eta_1 f(\eta_0)\,\chi(\eta) = 0 \tag{3.12}$$

for every continuous $f(\eta_0)$. We choose $f(\eta_0) = \eta_0$, and thus write

$$\int_0^\infty d\eta_0 \int d\eta_1\, \eta_0\, \chi(\eta) = 0\,. \tag{3.13}$$

Now we rotate the coordinate system by an angle δ:

$$\left\{\begin{aligned} \eta_0' &= \quad \eta_0 \cos\delta + \eta_1 \sin\delta \\ \eta_1' &= -\eta_0 \sin\delta + \eta_1 \cos\delta\,, \end{aligned}\right. \tag{3.14}$$

Provided that δ is small enough, then $\eta_0' \geq 0$ still has a non-vanishing interval from $\mathfrak{B}$. Thus we have

$$\int_0^\infty d\eta_0' \int d\eta_1'\, \eta_0'\, \chi'(\eta') = 0\,, \tag{3.15}$$

where we have to set $\chi'(\eta') = \chi(\eta)$. In the old variables

$$\int d\eta_0 \int d\eta_1 (\eta_0 \cos\delta + \eta_1 \sin\delta)\,\chi(\eta) = 0\,, \qquad \eta_0 \cos\delta + \eta_1 \sin\delta > 0\,. \tag{3.16}$$

Now the region of integration $\eta_0 \cos\delta + \eta_1 \sin\delta > 0$ is divided into $\eta_0 > 0$, region I: $\eta_1 < 0$ and $-\eta_1 \operatorname{tg}\delta > \eta_0 > 0$, and region II: $\eta_1 > 0$ and $0 > \eta_0 > -\eta_1 \operatorname{tg}\delta$. Thus due to (13)

$$\begin{aligned} \sin\delta \int_0^\infty d\eta_0 \int d\eta_1\, \eta_1 \chi(\eta) - \int_{(\mathrm{I})} d\eta_0 \int d\eta_1 (\eta_0 \cos\delta + \eta_1 \sin\delta)\,\chi(\eta) \,+ \\ + \int d\eta_0 \int_{(\mathrm{II})} d\eta_1 (\eta_0 \cos\delta + \eta_1 \sin\delta)\,\chi(\eta) = 0\,. \end{aligned} \tag{3.17}$$

Let $\chi(\eta) = 0$ for $\eta_0^2 + \eta_1^2 \geq \mathrm{R}^2$ and $|\chi(\eta)| < \mathrm{A}$, then we can easily find the estimate

$$\int d\eta_0 \int_{(\mathrm{I})} d\eta_1 (\eta_0 \cos\delta + \eta_1 \sin\delta)\,\chi(\eta) \;< 2R^3 A\delta \sin\delta\,. \tag{3.18}$$

A corresponding relation is true for the integral over II. Now we divide (3.17) by δ and we let $\delta \to 0$. It follows that

$$\int_0^\infty d\eta_0 \int d\eta_1\, \eta_1\, \chi(\eta) = 0\,. \tag{3.19}$$

Since the coordinate system (excluding the mentioned restriction) was arbitrary, it follows that with $\chi(\eta)$ also $l(\eta)\chi(\eta)$, where $l(\eta)$ is an arbitrary linear polynomial, has the property to yield zero upon integration over every half-plane which does not contain $\mathfrak{B}$. Due to the continuity of $l(\eta)\chi(\eta)$ we can integrate over every straight line which does not meet $\mathfrak{B}$ and obtain zero. $l(\eta)\chi(\eta)$ therefore fulfills the same assumptions as $\chi(\eta)$. Making use of induction we then obtain

$$\int_0^\infty d\eta_0 \int d\eta_1 \, P(\eta_0, \eta_1)\, \chi(\eta) = 0 \,, \tag{3.20}$$

where $P(\eta_0, \eta_1)$ is an arbitrary polynomial. (3.20) is valid in every half-plane which does not contain $\mathfrak{B}$. Due to the completeness of the polynomial, $\chi(\eta)$ vanishes in every such half-plane. Since $\mathfrak{B}$ is convex, $\chi(\eta) = 0$ for $\eta \notin \mathfrak{B}$.

The proof for the n-dimensional case clearly is obtained in the same fashion. In place of the single transformation (3.14) we have to consider (n − 1):

$$\left\{\begin{array}{ll} \left.\begin{array}{l} \eta_0' = \eta_0 \cos\delta + \eta_k \sin\delta \\ \eta_k' = -\eta_0 \sin\delta + \eta_k \cos\delta \end{array}\right\} & k = 1, 2, \ldots, n-1 \\ \eta_l' = \eta_l & \qquad l \neq k,\ l \neq 0 \end{array}\right. \tag{3.21}$$

(3.18) changes in an unimportant way. In place of (3.19) we have

$$\int_0^\infty d\eta_0 \int d^{n-1}\eta \, \eta \;\; \chi(\eta) = 0 \tag{3.22}$$

and in place of (3.20) a n-fold integral which contains an arbitrary polynomial $P(\eta_0, \eta_1, \ldots, \eta_{n-1})$.

As an application of Theorem 1 we shall determine the functions $\hat{\Phi}(\kappa^2, \mathbf{u})$, which belongs to the problem as presented in Section 1. As mentioned in Section 2, the corresponding $F(q_0, \mathbf{q})$ disappears for all double cones $q_0| + |\mathbf{q} - \mathbf{a}| < |\mathbf{a}| - 1$. Therefore, $\hat{\Phi}(\kappa^2, \mathbf{u})$ disappears in all cones $\kappa^2 + |\mathbf{u} - \mathbf{a}|^2 < (|\mathbf{a}| - 1)^2$, $|\mathbf{a}| > 1$. $\hat{\Phi}(\kappa^2, \mathbf{u})$ is different from zero at most in the cylinder $|\mathbf{u}| \leq 1$. As a conclusion, from this we obtain that $L(\xi;\eta)$, according to (3.1) and (2.4), is a regular analytic function in η. With this the Ansatz (2.2) is quite adequately justified.

Furthermore, we note that it is readily possible to extend Theorm 1 to general regions, since the Auxiliary Theorem used in the proof is formulated for arbitrary finite convex regions.

In regard to applications we give the following theorem which can be obtained from elementary considerations:[7]

Theorem 2: Necessary and sufficient conditions for $\tilde{F}(q_0, \mathbf{q})$ to vanish outside

$$(|q_0| + a)^2 \geqslant \mathbf{q}^2 + b^2 \qquad\qquad (a, b \geqslant 0)$$

are that $\hat{\Phi}(\kappa^2, \mathbf{u})$ be different from zero only in

$$|\boldsymbol{u}| \leqslant a$$

and

$$\varkappa \geqslant \mathrm{Max}\,\{0;\ b - \sqrt{a^2 - \boldsymbol{u}^2}\}$$

where ($\kappa = +\sqrt{\kappa^2}$).

4. THE FUNCTION $\widetilde{F}_R(k_0, \mathbf{k})$

In case $\hat{\Phi}(\kappa^2, \mathbf{u})$ in (3.2) diminishes strongly enough as $\kappa^2 \to \infty$, then there exists

$$\widetilde{F}_R(k_0\ \boldsymbol{k}) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{F(q_0, \boldsymbol{k})}{q_0 - h_0}\, \mathrm{d}q_0 \,. \tag{4.1}$$

for $\mathrm{Im}\,[k_0] > 0$ and represents for $\mathrm{Im}[k_0] = +0$ the Fourier-transform of $F_R(\xi_0, \boldsymbol{\xi})$.

Substitution of (3.2) yields

$$\widetilde{F}_R(k_0, \boldsymbol{k}) = \int \mathrm{d}^3u\, \hat{\Phi}_R(k_0^2 - (\boldsymbol{k} - \boldsymbol{u})^2, \boldsymbol{u}) \tag{4.2}$$

with

$$\hat{\Phi}_R(\varkappa^2, \boldsymbol{u}) = \frac{1}{2\pi i}\int\limits_0^{\infty} \mathrm{d}\lambda^2\, \frac{\hat{\Phi}(\lambda^2, \boldsymbol{u})}{\lambda^2 - \varkappa^2}\,. \tag{4.3}$$

$\hat{\Phi}_R(\kappa^2, \mathbf{u})$ is regularly analytic in κ^2 complex plane which is cut along the positive real axis from 0 to ∞. From this follows that $\widetilde{F}_R(k_0, \mathbf{k})$ is regularly analytic if $\mathrm{Im}[k]$ lies in the future cone. This property is characteristic for the Fourier-transform of a retarded distribution.

Now from (4.3) follows

$$\hat{\Phi}_R(\varkappa^2, \boldsymbol{u}) = \frac{(\varkappa^2 + 1)^m}{2\pi i}\int\limits_0^{\infty} \mathrm{d}\lambda^2\, \frac{\hat{\Phi}(\lambda^2, u)}{(\lambda^2 + 1)^m(\lambda^2 - \varkappa^2)} + \mathfrak{R}_{m-1}(\varkappa^2)\,, \tag{4.4}$$

where $\mathfrak{R}_{m-1}(\kappa^2)$ represents a polynomial of degree $(m - 1)$ with coefficients which are functions of $\mathbf{u}$. Now, for a given distribution Φ, we can always choose an m such that (4.4) is meaningful. If $\widetilde{F}_R(k_0, \mathbf{k})$ exists at all, it must necessarily be of the form

$$\widetilde{F}_R(k_0, \boldsymbol{k}) = \frac{1}{2\pi i}\int \mathrm{d}^3u \int \mathrm{d}\lambda^2\, \frac{[k_0^2 - (\boldsymbol{k} - \boldsymbol{u})^2 + 1]^m\, \hat{\Phi}(\lambda^2, \boldsymbol{u})}{(\lambda^2 + 1)^m[\lambda^2 - k_0^2 + (\boldsymbol{k} - \boldsymbol{u})^2]} + \mathfrak{R}_{2(\,,-1)}(k_0, \boldsymbol{k})\,, \tag{4.5}$$

where $\mathfrak{R}_{2(m-1)}(k_0, \mathbf{k})$ represents a polynomial of degree $2(m - 1)$ in k_0, k_1, k_3, k_0.

If $\widetilde{F}(q_0,,q)$ vanishes in the double cone, and so Φ is zero in $\kappa^2 + u^2 < 1$, we readily obtain from (4.5) regular regions of $\widetilde{F}_R(h_0, h)$. The following, for instance, is true:

Theorem 3: If $\widetilde{F}(q)$ vanishes in $|q_0| + |q| < 1$ and if $\widetilde{F}_R(k)$ exists, then for

$$\mu^2 < \mathrm{Max}\,[0;\ (1 - |q|^2 - q_0^2] \tag{4.6}$$

it is regularly analytic. Here $\mu = \mathrm{Im}[k]$ and $q = \mathrm{Re}[k]$.

Conclusion:[8] $\widetilde{F}_R(k)$ is regular in $|k_\nu| < \frac{1}{2}(\sqrt{3} - 1)$.

Proof: According to (4.5), $\widetilde{F}_R(k_0, k)$ is regular so long as $k_0^2 - \lambda^2 - (k - u)^2$ does not vanish for $\lambda^2 + u^2 \geq 1$. If one separates real and imaginary parts, then we must have

$$q_0^2 + \mu^2 < \lambda^2 + (q - u)^2 \tag{4.7}$$

as soon as $\lambda^2 + u^2 \geq 1$ and $\mu_0 q_0 - \mu(q - u) = 0$.

Now, however, for a fixed q the minimum of $\lambda^2 + (q - u)^2$ is equal to $(1 - |q|)^2$. Due to this, $\widetilde{F}_R(k)$ is regular for $\mu^2 < (1 - |q|)^2 - q_0^2$. Otherwise, $\widetilde{F}_R(k)$ is trivially regular for $\mu^2 < 0$. Naturally, the theorem is valid also for even $\widetilde{F}(q)$, as follows from Section 2. (Appendix)

For the derivation of the conclusion we write $k_\nu = \rho_\nu \exp[i\varphi_\nu]$; then (4.6) means

$$\sum_{k=1}^{3} \varrho_k^2 \sin^2\varphi_k + \varrho_0^2 \cos^2\varphi_0 < [1 - (\sum_1^3 \varrho_k^2 \cos^2\varphi_k)^{\frac{1}{2}}]^2 + \varrho_0^2 \sin^2\varphi_0\,, \tag{4.8}$$

or

$$\sum_0^3 \varrho_\nu^2 < \tfrac{1}{2} + 2[\tfrac{1}{2} - (\sum_1^3 \varrho_k^2 \cos^2\varphi_k)^{\frac{1}{2}}]^2 + 2\varrho_0^2 \sin^2\varphi_0\,. \tag{4.9}$$

If we want an idea of the kind of conclusion obtained, we set $\rho_\nu = \rho$ and $\varphi_\nu = 0$. The resulting quadratic inequality leads to the conclusion.

APPENDIX

Let

$$F(\xi_0, \xi) = F_R(\xi_0, \xi) + F_R(-\xi_0, \xi)\,. \tag{A.1}$$

The existence of

$$G(\xi_0, \xi) = G_R(\xi_0, \xi) - G_R(-\xi_0, \xi) \tag{A.2}$$

is asserted in such a way that $F(\xi) = (\partial/\partial\xi_0)G(\xi)$. We would like an additional relation, so we set

$$F_R(\xi) = \frac{\partial G_R(\xi)}{\partial \xi_0} \tag{A.3}$$

(A.3) means[9]

$$F_R(\varphi) = -G_R\left(\frac{\partial\varphi}{\partial\xi_0}\right), \tag{A.4}$$

whereby $G_R(\psi)$ is defined for all ψ for which $\int_{-\infty}^{\infty}\psi d\xi_0 = 0$. Now let $\alpha(\xi_0) \in S_{\xi_0}$ vanish in $\xi_0 > -1$ and satisfy $\int\alpha(\xi_0)d\xi_0 = 1$.

An arbitrary ψ will now be represented in

$$\psi(\xi_0, \boldsymbol{\xi}) = \chi(\xi_0, \boldsymbol{\xi}) + \alpha(\xi_0)\int_{-\infty}^{\infty}\psi(\tau, \boldsymbol{\xi})\, d\tau\,. \tag{A.5}$$

Here

$$\int_{-\infty}^{\infty}\chi(\xi)\, d\xi_0 = 0\,,$$

is

$$\varphi(\xi_0, \boldsymbol{\xi}) = \int_{-\infty}^{\xi_0}\chi(\tau, \boldsymbol{\xi})\, d\tau \in \mathcal{S}\,. \tag{A.6}$$

Now we define

$$G_R(\psi) = -F_R(\varphi) \tag{A.7}$$

and $G(\xi_0, \boldsymbol{\xi})$ by (A.2)

G_R disappears outside the future cone. Indeed, if ψ has some support outside the future cone then the same is valid for χ since it follows from $\alpha(\xi_0)$. But the statement is valid also for φ, since every plane $\boldsymbol{\xi}$ = a constant has only one half-plane in common with the future cone.

Thus $G_R(\psi) = 0$.

Finally, $\tilde{F}(q)$ and $\tilde{G}(q)$ have the same support, then we have

$$\tilde{F}(\varphi) = -i\tilde{G}(p_0\varphi)\,, \tag{A.8}$$

whereby $\tilde{G}(\psi)$ is defined for odd ψ. But for even ψ we have $\tilde{G}(\psi) = 0$.

REFERENCES

1. This condition (b) will be made more distinct in many cases. This is done when—in a certain model—it is assumed that (except for the vacuum state) the smallest eigenvalue of the operator P^2 is the rest mass squared of the lightest particle and is different from zero. This case also can be easily treated with our method.
2. We always speak of smooth distributions. L. Schwartz: Theorie des Distributions—(Paris, 1951) Vol. II, p. 93.

3.

$$\int \varphi(\xi) L(\xi; \boldsymbol{\eta}) \mathrm{d}^4\xi \in O_M$$

for every test function $\varphi(\xi) \in S\xi$. L. Schwartz, l.c., p. 99.

4. L. Schwartz: l.c., p. 100.

5. Similarly if we write

$$L(\xi; \boldsymbol{\eta}) = \varepsilon(\xi_0)\, \Phi(\xi_0^2 - \boldsymbol{\xi}^2, \boldsymbol{\eta}),$$

then we have formally

$$\overset{\wedge}{\Phi}(\varkappa^2, \boldsymbol{u}) = \frac{4i}{(2\pi)} \frac{\partial}{\partial \varkappa^2} \theta(\varkappa^2) \int \mathrm{d}\alpha^2 \int \mathrm{d}^3\boldsymbol{\eta} \exp[-i\boldsymbol{u}\boldsymbol{\eta}] J_0(\alpha\varkappa)\, \Phi(\alpha^2, \boldsymbol{\eta})$$

where J_0 is the Bessel function of order zero.

$$\bar{\Delta}(\varkappa^2, x_0^2 - \boldsymbol{x}^2) = \frac{1}{4\pi} \frac{\partial}{\partial \varkappa^2} \theta(\varkappa^2) J_0(\varkappa \sqrt{x_0^2 - \boldsymbol{x}^2})$$

is the known invariant function.

6. We are indebted for this proof to Mr. F. J. Dyson to whom we also wish to express our thanks here. He replaces the original, not the elementary proof of the authors.

7. The choice of the region in which $F(q_0, q)$ vanishes corresponds to the distinct spectrum condition mentioned in reference 1.

8. This conclusion corresponds to a theorem which was communicated by N. N. Bogoliubov at the International Congress on Theoretical Physics in Seattle (September 1956). We would like to thank Mr. N. N. Bogoliubov for stimulating discussions.

9. L. Schwartz: l.c., Vol. I. Ch. II. Test functions always from S (Vol. II, pg. 89)

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Integral Representations of Causal Commutators

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An integral representation is found for the matrix element, between given states, of the commutator of two field operators. The representation makes use of the information derivable from the local commutativity of the operators and from the mass spectrum of the fields. The representation was discovered by Jost and Lehmann and proved by them for the case of two fields of equal mass. It is here extended to the case of unequal masses.

The mathematical basis of this work is the fact that every function $f(q)$ of a four-vector q, with a Fourier transform $\tilde{f}(x)$ which vanishes for space like x, has a unique extension which is a solution of the wave equation in six dimensions.

I. STATEMENT OF THE PROBLEM

JOST and Lehmann[1] have introduced an integral representation for matrix elements of causal commutators. Their representation is a powerful tool for investigating the analytic behavior of scattering amplitudes as a function of all relevant momentum and energy variables. In particular, Lehmann[2] was able to obtain a proof of dispersion relations, shorter and simpler than the original proof of Bogolyubov.[3] Unfortunately, the Jost-Lehmann analysis applied only to the commutator of two fields of equal mass. The Lehmann proof of dispersion relations was restricted to the scattering of equal-mass particles, and for this reason remained unpublished. In this paper we extend the integral representation to the case of unequal masses.

A more important task for the future is to explore systematically the analytic consequences of causality, not restricting attention to the special type of information which can be embodied in dispersion relations. Progress in this direction has already been made by Källén and Wightman,[4] using only the causality condition and assuming nothing about particle masses. We hope that the integral representations here established may make it possible to extend the Källén-Wightman analysis so as to include detailed information about the mass spectrum.

We are concerned with two fields $A(x)$ and $B(x)$, obeying the causality principle

$$[A(x),B(x')]=0 \quad \text{for} \quad (x-x')^2<0. \tag{1}$$

Here x and x' are 4-vectors, and the scalar product is defined by

$$x\cdot y=x_0y_0-\mathbf{x}\cdot\mathbf{y}=x_0y_0-x_1y_1-x_2y_2-x_3y_3. \tag{2}$$

The problem is to determine the analytic form of the function

$$\tilde{f}(x)=i\langle P,\alpha|[A(\tfrac{1}{2}x), B(-\tfrac{1}{2}x)]|Q,\beta\rangle, \tag{3}$$

or of its Fourier transform

$$f(q)=\frac{1}{(2\pi)^4}\int d_4x\exp(-iq\cdot x)\tilde{f}(x). \tag{4}$$

Here $|P,\alpha\rangle$ and $|Q,\beta\rangle$ are any two states specified by the energy-momentum vectors P and Q and by other quantum numbers α and β.

By Eq. (1), the function $\tilde{f}$ in x space has the property

$$\tilde{f}(x)=0 \quad \text{for} \quad x^2<0. \tag{5}$$

From Eqs. (3) and (4), the function f in q space also has the property of vanishing in a large region. The two terms of the commutator give

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[1] R. Jost and H. Lehmann, Nuovo cimento 5, 1598 (1957).

[2] H. Lehmann (private communication). *Note added in proof.*—Lehmann has used the results of this paper to prove that certain scattering amplitudes are analytic functions of the momentum transfer (to be published in Nuovo cimento).

[3] Bogolyubov, Medvedev, and Polivanov, Uspekhi Math. Nauk (to be published).

[4] G. Källén and A. S. Wightman (private communication to W. Pauli). See report by G. Källén in *Proceedings of Seventh Annual Rochester Conference on High-Energy Nuclear Physics, 1957* (Interscience Publishers, New York, 1957, Session IV, p. 17.

$$f(q)=f_1(q)-f_2(q), \tag{6}$$

where $f_1(q)=0$ unless $[\frac{1}{2}(P+Q)-q]$ is the energy-momentum vector of a state $|n\rangle$ for which

$$\langle P,\alpha|A(0)|n\rangle\neq 0, \quad \langle n|B(0)|Q,\beta\rangle\neq 0, \tag{7}$$

and $f_2(q)=0$ unless $[\frac{1}{2}(P+Q)+q]$ is the momentum vector of a state with

$$\langle P,\alpha|B(0)|n\rangle\neq 0, \quad \langle n|A(0)|Q,\beta\rangle\neq 0. \tag{8}$$

Suppose that all states $|n\rangle$ for which Eq. (7) holds have mass not less than b, and that all for which Eq. (8) holds have mass not less than c. The values of b and c depend mainly on selection rules restricting the numbers of particles of various types which can be present in the state $|n\rangle$. So long as $A(x)$ and $B(x)$ are distinct fields, the values of the masses b and c will in general be different. Without loss of generality we may choose a coordinate system in which

$$\tfrac{1}{2}(P+Q)=[a,0,0,0], \quad a>0. \tag{9}$$

Then from Eq. (6) we see that $f(q)=0$ for all $q=(q_0,\mathbf{q})$ satisfying

$$a-[\mathbf{q}^2+b^2]^{\frac{1}{2}}<q_0<[\mathbf{q}^2+c^2]^{\frac{1}{2}}-a. \tag{10}$$

Theorem 2 in the paper of Jost and Lehmann[1] runs as follows.

Theorem A (*Jost-Lehmann*).—The class of functions $f(q)$ which vanish in the region

$$|q_0|<[\mathbf{q}^2+c^2]^{\frac{1}{2}}-a, \tag{11}$$

and have Fourier transforms $\tilde{f}(x)$ vanishing for $x^2<0$, is identical with the class of functions of the form

$$f(q)=\int d_3u\int dk^2\, \epsilon(q_0)\delta[q_0^2-(\mathbf{q}-\mathbf{u})^2-k^2] \times[\Phi_1(\mathbf{u},k^2)+q_0\Phi_2(\mathbf{u},k^2)], \tag{12}$$

with the functions $\Phi_1(\mathbf{u},k^2)$ and $\Phi_2(\mathbf{u},k^2)$ each vanishing except in the region

$$|\mathbf{u}|\leqslant a, \quad k\geqslant c-(a^2-\mathbf{u}^2)^{\frac{1}{2}}. \tag{13}$$

This theorem provides a representation for functions $f(q)$ vanishing in the region (10), only in the symmetric case $b=c$. The symmetry in q_0 played an essential part in the Jost-Lehmann proof. One cannot expect a representation like Eq. (12), with the parts even and odd in q_0 specified by independent functions Φ_2 and Φ_1, to hold for an unsymmetric region. The analog of Eq. (12) in the unsymmetric case must treat the four components (q_0,q_1,q_2,q_3) on an equal footing. The difficulty in extending the representation to unsymmetric regions arises precisely from the fact that in the symmetric case Eq. (12) has a somewhat accidental and misleading simplicity.

For later reference we also state, as a lemma, Theorem 1 of Jost and Lehmann.[1]

Lemma 1 (*Jost-Lehmann*).—The class of functions $f(q)$ satisfying Eq. (5), and vanishing in the region

$$|q_0|+|\mathbf{q}-\mathbf{v}|<\lambda, \tag{14}$$

is identical with the class represented by Eq. (12) with the functions Φ_1 and Φ_2 vanishing in

$$k^2+(\mathbf{u}-\mathbf{v})^2<\lambda^2. \tag{15}$$

The deduction of Theorem A from the Lemma is immediate, since the region (11) is a union of regions (14), and the region (13) is the complement of the union of the corresponding regions (15).

2. REPRESENTATIONS ON A GENERAL SURFACE

We define the class C to consist of all functions $f(q)$ whose Fourier transforms $\tilde{f}(x)$ vanish for $x^2<0$. We require a representation for a function in C which also vanishes in some region R in q space. It is convenient to divide the problem into two parts. In this section we find representations for all functions of C. Restrictions on the behavior of $f(q)$ in q space will be considered in Sec. 3.

The key to the problem[5] is to extend the four-dimensional x and q spaces to six dimensions. The two additional dimensions are to be spacelike. Let z denote the six vector $(z_0=x_0,\ z_1=x_1,\ z_2=x_2,\ z_3=x_3,\ z_4=y_1,\ z_5=y_2)$, and r the six vector $(r_0=q_0,\ r_1=q_1,\ r_2=q_2,\ r_3=q_3,\ r_4=p_1,\ r_5=p_2)$. The metric is defined by

$$z^2=x^2-y^2=x_0^2-x_1^2-x_2^2-x_3^2-y_1^2-y_2^2. \tag{16}$$

The special six-vector $(q_0,q_1,q_2,q_3,0,0)$ will be denoted by $\hat{q}$. In general, Greek suffixes will be summed from 0 to 5, Latin suffixes from 0 to 3.

By Eq. (5), the six-dimensional function

$$\tilde{F}(z)=4\pi\tilde{f}(x)\delta(x^2-y^2)=4\pi\tilde{f}(x)\delta(z^2) \tag{17}$$

determines the four-dimensional $\tilde{f}(x)$; in fact,

$$\tilde{f}(x)=\frac{1}{4\pi}\int_0^\infty \tilde{F}(z)dy^2=\frac{1}{4\pi^2}\int \tilde{F}(z)d_2y. \tag{18}$$

Taking the Fourier transform of Eq. (17), we obtain

$$F(r)=\frac{1}{(2\pi)^6}\int e^{-i(r\cdot z)}F(z)d_6z$$

$$=\int f(q)D^1(r-\hat{q})d_4q, \tag{19}$$

where $D^1(r)$ is the even invariant function in six dimensions,

$$D^1(r)=\frac{2}{(2\pi)^5}\int e^{-i(r\cdot z)}\delta(z^2)d_6z=\frac{1}{\pi^3(r^2)^2}. \tag{20}$$

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[5] For the idea of going into six dimensions I am indebted to Dr. Jost.

The functions $D^1(r)$ and $F(r)$ are solutions of the wave equation

$$\Box_6 D^1(r)=0, \quad \Box_6 F(r)=0, \tag{21}$$

$$\Box_6=\frac{\partial^2}{\partial r_0^2}-\frac{\partial^2}{\partial r_1^2}-\frac{\partial^2}{\partial r_2^2}-\frac{\partial^2}{\partial r_3^2}-\frac{\partial^2}{\partial r_4^2}-\frac{\partial^2}{\partial r_5^2}. \tag{22}$$

From Eqs. (4), (18) and (19), we have

$$f(q)=F(\hat{q}), \tag{23}$$

so that $F(r)$ is a simple extrapolation of the function $f(q)$ from four to six dimensions.

As a solution of the wave equation, $F(r)$ can be expressed in terms of its value and its normal derivative on an arbitrary spacelike surface Σ in r space. In fact,

$$F(r)=\int d\Sigma_\alpha\left[F(r'),\frac{\partial}{\partial r_\alpha'}D(r'-r)\right], \tag{24}$$

where the bracket notation is defined by

$$\left[F,\frac{\partial}{\partial r_\alpha'}D\right]=F\frac{\partial D}{\partial r_\alpha'}-D\frac{\partial F}{\partial r_\alpha'}, \tag{25}$$

the surface element $d\Sigma_\alpha$ is a six-vector normal to the surface in the hyperbolic metric, and $D(r)$ is the odd invariant function in six dimensions,

$$D(r)=(1/2\pi^2)\epsilon(r_0)\delta'(r^2). \tag{26}$$

In particular, Eqs. (23) and (24) give

$$f(q)=\int d\Sigma_\alpha\left[F(r),\frac{\partial}{\partial r_\alpha}D(r-\hat{q})\right], \tag{27}$$

which is the desired representation of $f(q)$ on a general surface Σ.

Suppose that Σ is a special surface independent of the (r_4,r_5) coordinates. We then write

$$(r_0,r_1,r_2,r_3,r_4,r_5)=(u_0,u_1,u_2,u_3,p_1,p_2), \tag{28}$$

$$s=p_1^2+p_2^2, \tag{29}$$

and regard Σ as a surface in the four-dimensional u space. Equation (27) reduces to

$$f(q)=\frac{1}{2\pi}\int d\Sigma_j$$

$$\times\int_0^\infty ds\left[F(u,s),\frac{\partial}{\partial u_j}\epsilon(u_0-q_0)\delta'((u-q)^2-s)\right], \tag{30}$$

with $F(u,s)$ given by Eq. (19),

$$F(u,s)=\frac{1}{\pi^2}\int f(q)\frac{1}{[(u-q)^2-s]^2}d_4q. \tag{31}$$

As a still more special case, let Σ be the surface $u_0=0$. Then Eq. (30) becomes identical with Eq. (12), provided we take

$$\left.\begin{aligned}\Phi_1(\mathbf{u},s)&=-\frac{1}{\pi}\frac{d}{ds}[\theta(s)F_1(\mathbf{u},s)],\\ \Phi_2(\mathbf{u},s)&=-\frac{1}{\pi}\frac{d^2}{ds^2}[\theta(s)F_2(\mathbf{u},s)],\end{aligned}\right\} \tag{32}$$

$$F_1(\mathbf{u},s)=\frac{1}{\pi^3}\int f(q)\frac{2q_0}{[q_0^2-(\mathbf{u}-\mathbf{q})^2-s]^3}d_4q, \tag{33}$$

$$F_2(\mathbf{u},s)=\frac{1}{\pi^3}\int f(q)\frac{1}{[q_0^2-(\mathbf{u}-\mathbf{q})^2-s]^2}d_4q. \tag{34}$$

All the integrals containing singular kernels are to be interpreted in the distribution-theory sense,[6] i.e., they are to be integrated by parts repeatedly until the singularity disappears.

We now prove the uniqueness of the representation (27). By this we mean that if a function $f(q)$ in the class C is given, and the surface Σ in r space, and if $f(q)$ has a representation (27) with any function $F(r)=F(u,s)$ which depends on (r_4,r_5) only in the combination s, and if $F(r)$ satisfies the wave equation (21), then F must be identical with the function defined by Eq. (19). To prove the uniqueness, observe first that for any $F(r)$ satisfying the wave equation, the right side of Eq. (27) is independent of Σ. Suppose if possible that there exists a Σ, and a function $F(r)$ satisfying the stated conditions, such that the right side of Eq. (27) vanishes for all q. Then the right side of Eq. (27) vanishes for all Σ, and in particular for the special surface $u_0=0$. Therefore the right side of Eq. (12) vanishes, when Φ_1 and Φ_2 are defined by Eq. (32), and $F_1(\mathbf{u},s)$ and $F_2(\mathbf{u},s)$ are the normal derivative and value of $F(u,s)$ on $u_0=0$. But the uniqueness of the representation (12) was already proved by Jost and Lehmann.[1] Therefore Φ_1 and Φ_2 must be zero, and by integrating Eq. (32) from $s=0$ it follows that $F_1=F_2=0$. But $F(u,s)$ satisfies the wave equation, and the vanishing of F and of its normal derivative on $u_0=0$ implies that $F=0$ everywhere.

The conclusion of this section is that there is a one-to-one correspondence between functions $f(q)$ of the class C and solutions $F(r)$ of the wave equation in six dimensions with rotational symmetry in the (45) plane. Each $f(q)$ has a representation (27) in which $F(r)$ appears as a weight function. And the representation is unique, the weight function being determined by Eq. (19) when $f(q)$ is given.

3. GENERAL UNSYMMETRIC REGIONS

We take as the widest convenient generalization of the region (10) a region R bounded by any two spacelike surfaces σ_1, σ_2 in q space. R is defined by

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[6] L. Schwartz, *Théorie des Distributions* (Hermann et Cie, Paris, 1951), Vol. 1, Chap. 2.

$$f(\mathbf{q})<q_0<g(\mathbf{q}), \tag{35}$$

where f and g are functions satisfying

$$|f(\mathbf{q})-f(\mathbf{q}')|\leqslant|\mathbf{q}-\mathbf{q}'|,\quad |g(\mathbf{q})-g(\mathbf{q}')|\leqslant|\mathbf{q}-\mathbf{q}'|. \tag{36}$$

We consider the class C_R of functions $f(q)$ which belong to C and also vanish in R.

We call a hyperboloid

$$(q-u)^2-s=0,\quad s\geqslant 0, \tag{37}$$

admissible if its upper sheet does not come below σ_2 and its lower sheet does not come above σ_1. Such hyperboloids correspond to points $r=(u_0,u_1,u_2,u_3,p_1,p_2)$, $s=p_1^2+p_2^2$, lying in a certain region S of r space. For every r in S, the function $D(r-\hat{q})$ defined by Eq. (26) is zero for all q in R. This suggests that we look for a representation of $f(q)$ in C_R by an integral (27) with the points r confined to S. Every integral (27) with r in S belongs to C_R. Our main result will be that the converse is also true, so that every $f(q)$ in C_R has a representation (27) using only admissible hyperboloids.

The region S is defined by

$$m(\mathbf{u},s)\leqslant u_0\leqslant M(\mathbf{u},s),\quad s=p_1^2+p_2^2, \tag{38}$$

with

$$m(\mathbf{u},s)=\max_{\mathbf{q}}\{g(\mathbf{q})-[(\mathbf{q}-\mathbf{u})^2+s]^{\frac{1}{2}}\}, \tag{39}$$

$$M(\mathbf{u},s)=\min_{\mathbf{q}}\{f(\mathbf{q})+[(\mathbf{q}-\mathbf{u})^2+s]^{\frac{1}{2}}\}. \tag{40}$$

The right sides of Eqs. (39) and (40) are necessarily finite because of Eq. (36). S is bounded by two surfaces Σ_1, Σ_2 in r space. Since these surfaces are envelopes of two families of hyperboloids, they are also spacelike. Define the region T by

$$M(\mathbf{u},s)<u_0<m(\mathbf{u},s),\quad s=p_1^2+p_2^2. \tag{41}$$

For r to belong to T, the hyperboloid (37) must be doubly inadmissible, i.e., its upper sheet must come below σ_2 and its lower sheet above σ_1. The decisive step in our argument is contained in the following:

Lemma 2.—The function $F(r)=0$ for each point r in T.

In proving the lemma, we may choose the origin of coordinates in q space so that the point r is $(0,0,0,0,p_1,p_2)$, with $t=p_1^2+p_2^2$. The hyperboloid $q^2=t$ is doubly inadmissible, and we are required to prove that $F(0,t)=0$.

There exist two points q_1 and q_2 such that q_1 lies above σ_1 and below the lower sheet of $q^2=t$, while q_2 lies below σ_2 and above the upper sheet. Without loss of generality we may assume also

$$q_1^2=q_2^2=\lambda^2>t. \tag{42}$$

By a Lorentz transformation in the q space we can choose a coordinate system in which

$$\mathbf{q}_1=\mathbf{q}_2=\mathbf{v},\quad q_{20}=q_{10}=(\mathbf{v}^2+\lambda^2)^{\frac{1}{2}}. \tag{43}$$

In this coordinate system we represent $f(q)$ in the Lehmann-Jost form (12). Since q_1 is above σ_1 and q_2 below σ_2, the whole of the double cone

$$|q_0|+|\mathbf{q}-\mathbf{v}|\leqslant(\mathbf{v}^2+\lambda^2)^{\frac{1}{2}} \tag{44}$$

is interior to R. Therefore $f(q)$ vanishes in the region (44), and Lemma 1 states that $\Phi_2(\mathbf{u},s)$ vanishes for

$$s+(\mathbf{u}-\mathbf{v})^2<\mathbf{v}^2+\lambda^2. \tag{45}$$

But $\Phi_2(\mathbf{u},s)$ is related to $F_2(\mathbf{u},s)$ by Eq. (32), and $F_2(\mathbf{u},s)$ is the value of $F(u,s)$ on $u_0=0$. Therefore $F(u,s)$ vanishes for

$$u_0=0,\quad s+(\mathbf{u}-\mathbf{v})^2<\mathbf{v}^2+\lambda^2, \tag{46}$$

and by Eq. (42) this implies in particular that $F(0,t)=0$.

Consider the representation (27) of the function $f(q)$. We choose for Σ the surface

$$u_0=\tfrac{1}{2}[m(\mathbf{u},s)+M(\mathbf{u},s)],\quad s=p_1^2+p_2^2, \tag{47}$$

with m and M given by Eqs. (39) and (40). The precise position of Σ is unimportant; we might use any Σ which is spacelike and lies everywhere between the surfaces Σ_1 and Σ_2. The essential point is that, by Eqs. (38) and (41), every point of Σ belongs either to S or to T. But we have proved that $F(r)=0$ in T. Therefore

$$f(q)=\int_S d\Sigma_\alpha\left[F(r),\frac{\partial}{\partial r_\alpha}D(r-\hat{q})\right], \tag{48}$$

where the integral extends only over the part of Σ included in S. We summarize this result in the following:

Theorem B.—A function $f(q)$ belongs to the class C_R if and only if it is representable in the form (48), with the surface Σ defined by Eq. (47) and the set S by Eqs. (38), (39), and (40).

In the proof of Theorem B, it was essential to choose a definite surface Σ and to use the uniqueness of the representation on Σ in order to remove from the integration the part of Σ lying outside S. For applications of the theorem, the specification of Σ inside S is an unnecessary encumbrance and the uniqueness is unimportant. The following theorem, while apparently weaker than Theorem B, is for practical purposes equivalent to it.

Theorem C.—For a function $f(q)$ to vanish in the region (35) and to have a Fourier transform vanishing for $x^2<0$, it is necessary and sufficient that

$$f(q)=\int d_4u\int ds\ \epsilon(q_0-u_0)\delta[(q-u)^2-s]\Psi(u,s), \tag{49}$$

where $\Psi(u,s)$ is a function vanishing outside the region S defined by Eqs. (38), (39), and (40).

The representation (48) is a special case of (49), but every representation (49) can be projected back to the form (48) by using Eq. (27).

Theorem C substantiates our earlier statement that every function $f(q)$ vanishing in R has a representation by means of admissible hyperboloids.[7]

4. SPECIAL UNSYMMETRIC REGIONS

We now specialize the conclusions of Sec. 3 to the case of the unsymmetric region (10). The calculations are elementary and we give only the results. Without loss of generality we assume $0\leqslant b\leqslant c$.

The region S defined by Eq. (38) is now

$$[\mathbf{u}^2+(c-k)^2]^{\frac{1}{2}}-a\leqslant u_0\leqslant a-[\mathbf{u}^2+(b-k)^2]^{\frac{1}{2}},\quad 0\leqslant k\leqslant b, \tag{50}$$

$$[\mathbf{u}^2+(c-k)^2]^{\frac{1}{2}}-a\leqslant u_0\leqslant a-|\mathbf{u}|,\quad b\leqslant k\leqslant c, \tag{51}$$

$$|\mathbf{u}|-a\leqslant u_0\leqslant a-|\mathbf{u}|,\quad k\geqslant c, \tag{52}$$

where we have written

$$k=s^{\frac{1}{2}}=[p_1{}^2+p_2{}^2]^{\frac{1}{2}}. \tag{53}$$

The values of $\mathbf{u}$ and k allowed by Eqs. (50)–(52) are restricted by

$$|\mathbf{u}|\leqslant a,\quad |k|\geqslant K(\mathbf{u}), \tag{54}$$

where the function $K(\mathbf{u})$ is

$$K(\mathbf{u})=A-a[1-(\mathbf{u}^2/B^2)]^{\frac{1}{2}},\quad |\mathbf{u}|\leqslant (B^2/a), \tag{55}$$

$$K(\mathbf{u})=c-2[a(a-|\mathbf{u}|)]^{\frac{1}{2}},\quad (B^2/a)\leqslant |\mathbf{u}|\leqslant a, \tag{56}$$

$$A=\tfrac{1}{2}(b+c),\quad B^2=a^2-\tfrac{1}{4}(b-c)^2. \tag{57}$$

When B^2 is negative, $K(\mathbf{u})$ is given by Eq. (56) for all $|\mathbf{u}|\leqslant a$.

Theorem C states that every $f(q)$ of class C and vanishing in the region (10) is representable in the form

$$f(q)=\int d_4u\int dk^2\,\epsilon(q_0-u_0)\delta[(q-u)^2-k^2]\Psi(u,k^2), \tag{58}$$

with $\Psi(u,k^2)$ vanishing except for (u,k) satisfying Eqs. (50), (51), and (52). The allowed values of (u,k) also satisfy Eq. (54), which is a generalization of Eq. (13) to the unsymmetric case. Theorem C thus provides the complete solution to the problem stated in Sec. 1.

An interesting consequence of Theorem C arises in the case $B^2<0$. In this case neither Eq. (50) nor Eq. (51) can be satisfied for any (u,k) with $k<c-2a$, and it makes no difference to the region S if we give b any value from zero up to $c-2a$. Theorem C therefore implies the following:

Corollary.—Suppose a function $f(q)$ of class C vanishes in the region

$$a-[\mathbf{q}^2+b^2]^{\frac{1}{2}}<q_0<[\mathbf{q}^2+c^2]^{\frac{1}{2}}-a, \tag{59}$$

with $b<c-2a$. Then $f(q)$ vanishes in the larger region

$$a-[\mathbf{q}^2+(c-2a)^2]^{\frac{1}{2}}<q_0<[\mathbf{q}^2+c^2]^{\frac{1}{2}}-a. \tag{60}$$

The corollary can easily be generalized to regions of arbitrary shape. The effect of it is to ensure that, if a function $f(q)$ in C is known to vanish in a region R with a sharply re-entrant "nose," we can enlarge the region by rounding off the tip of the nose. That a statement of this kind should be valid was first conjectured by Jost (private communication).

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The author is deeply indebted to Dr. Jost and Dr. Lehmann for their persistent help and encouragement.

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[7] It is an interesting mathematical question (but irrelevant to the physical applications) whether any restriction at all on the shape of the region R is necessary. I conjecture that for any open point-set R, every function $f(q)$ of class C vanishing in R has a representation of the form of Eq. (49), with $\Psi(u,s)$ nonzero only when the hyperboloid $(q-u)^2=s$ is completely outside R. This conjecture would have the interesting consequence that, for every $f(q)$ of class C, the closure of the set of points q for which $f(q)\neq 0$ must be a union of hyperboloids. I am able to prove the conjecture only if R is a region bounded by two spacelike surfaces (the case treated in the text) or if R is a union of several such regions $R_1, R_2, \cdots$, with every pair of points in (R_i,R_j), $i\neq j$, separated by a timelike interval.

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Analytic Properties of Scattering Amplitudes as Functions of Momentum Transfer.

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Summary. — Scattering amplitudes are shown to have analytic properties as functions of momentum transfer. The partial wave expansions which define physical scattering amplitudes continue to converge for complex values of the scattering angle, and define uniquely the amplitudes appearing in the unphysical region of non-forward dispersion relations. The expansions converge for all values of momentum transfer for which dispersion relations have been proved.

1. – Introduction.

The purpose of this note is to derive some properties of scattering amplitudes which follow from causality in relativistic quantum theory. It will be shown that a scattering amplitude has—for fixed energy—analytic properties as a function of scattering angle or momentum transfer. This consequence of causality is distinct from the existence of dispersion relations (1-4) which express analytic properties of a scattering amplitude as a function of energy for fixed momentum transfer. However, our results are of interest mainly in connection with dispersion relations for non-forward scattering. They imply that for all values of momentum transfer for which dispersion relations have

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(1) M. L. Goldberger: *Phys. Rev.*, **99**, 979 (1955).

(2) K. Symanzik: *Phys. Rev.*, **100**, 743 (1957).

(3) N. Bogoliubov, B. Medvedev and M. Polivanov: *lecture notes*. Translated at the Institute for Advanced Study, (Princeton, 1957).

(4) H. J. Bremermann, R. Oehme and J. G. Taylor: *Phys. Rev.*, **109**, 2178 (1958). These papers contain numerous other references.

been established (3,4)—in π-N scattering this is the case for momentum transfer $< (\frac{2}{3}\cdot(2m+\mu)/(2m-\mu))^{\frac{1}{2}}\cdot 4\mu$—the so-called non-physical region is completely determined by the physical phase shifts via the partial wave expansion. This possibility has been suggested previously (5).

We disregard spins and consider the scattering of a charged particle of mass m (nucleon) with initial momentum p, final momentum p', and a neutral particle of mass $\mu < 2m$ (meson) with initial momentum k, final momentum k'. We assume in addition that these particles are not coupled to other particles of charge zero and mass $< 2\mu$ or of charge one and mass $< m+\mu$. Our results are valid also for the scattering of equal particles ($m = \mu$) under corresponding restrictions on the mass spectrum.

As is well known, several equivalent expressions for the scattering matrix in terms of Heisenberg operators may be given. With the notation

$$\langle p'k' \text{ out} | pk \text{ in}\rangle = \langle p'k' \text{ in} | pk \text{ in}\rangle + i(2\pi)\,\delta(p+k-p'-k')T$$

the amplitude T may be written as (6)

$$T = -\int \mathrm{d}^4x \exp\left[\frac{i(k+k')}{2}\,x\right]\langle p'|R'A\left(\frac{x}{2}\right)A\left(-\frac{x}{2}\right)|p\rangle\,, \tag{1}$$

or

$$T = -\int \mathrm{d}^4x \exp\left[\frac{i(k'-p')}{2}\,x\right]\langle 0|R'A\left(\frac{x}{2}\right)\psi\left(-\frac{x}{2}\right)|pk \text{ in}\rangle\,. \tag{2}$$

$A(x)$ and $\psi(x)$ are the meson and nucleon field operators. R' denotes a retarded commutator. For example,

$$R'A(x)\psi(y) = -\,i(\Box x - \mu^2)(\Box y - m^2)\,\theta(x-y)\,[A(x), \psi(y)]\,.$$

The state vectors refer to incoming or outgoing particles with definite momenta as indicated.

Eq. (1) is used in the derivation of dispersion relations. Eq. (2) on the other hand yields directly information about the scattering amplitude as a function of momentum transfer, since—in the center of mass system—this variable appears only in the exponential. However, more information is obtained in both cases by observing that from either (1) or (2) the following expression for the imaginary part of the amplitude may be derived,

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(5) M. L. Goldberger: *Proceedings of the Sixth Annual Rochester Conference*, (New York, 1956).

(6) H. Lehmann, K. Symanzik and W. Zimmermann, *Nuovo Cimento*, **6**, 319 (1957).

$$(3)\qquad \operatorname{Im} T = \pi \int \mathrm{d}^4x_1\,\mathrm{d}^4x_2 \exp\left[\frac{i(k'-p')x_1}{2} - \frac{i(k-p)x_2}{2}\right]\cdot$$

$$\cdot \sum_\gamma \langle 0|R'A\left(\frac{x_1}{2}\right)\psi\left(-\frac{x_1}{2}\right)|p+k,\gamma\rangle\cdot\langle p+k,\gamma|R'A\left(\frac{x_2}{2}\right)\psi^+\left(-\frac{x_2}{2}\right)|0\rangle\,,$$

$\sum_\gamma$ denotes a sum over all states with total four-momentum $p+k$. Eq. (3) is (on the energy shell) equivalent to the unitary requirement for the scattering amplitude. It was first used by **Bogoliubov** *et al.* (3). A simple proof of this relation is given in Sect. **3**.

The general method used to obtain explicit consequences of these expressions may be described as follows:

In each case the scattering amplitude—or its imaginary part—appears as the Fourier transform of a retarded commutator or of a sum over products of such commutators. Therefore it is simply related to the Fourier-transform of the corresponding unretarded commutator which is in the case of (2) or (3) given by

$$(4)\qquad F(q) = \int \mathrm{d}^4x \exp[iqx]\langle 0|\left[j\left(\frac{x}{2}\right), f\left(-\frac{x}{2}\right)\right]|p+k,\gamma\rangle\,,$$

where $j(x) = (\square - \mu^2)A(x)$; $f(x) = (\square - m^2)\psi(x)$. We know about $F(q)$:

(4*a*) it is the Fourier transform of a function that vanishes for space-like x;

$F(q)$ vanishes unless

$$(4b)\qquad \left\{\begin{array}{lll} \dfrac{p_0+k_0}{2}+q_0 \geqslant 0 & \text{and} & \left(\dfrac{p+k}{2}+q\right)^2 \geqslant m_1^2\,, \\ \text{or} & & \\ \dfrac{p_0+k_0}{2}-q_0 \geqslant 0 & \text{and} & \left(\dfrac{p+k}{2}-q\right)^2 \geqslant m_2^2\,. \end{array}\right.$$

The latter statement follows directly if a sum over intermediate states is introduced in (4). m_1 and m_2 are the masses of the lowest intermediate states which contribute to the two terms of the commutator. In the π-N case $m_1 = 3\mu$, $m_2 = m+\mu$.

Dyson (⁷) has solved the problem of finding a representation of all functions satisfying these conditions. His result is: For $F(q)$ to satisfy (4*a*) and (4*b*) it is necessary and sufficient that it can be represented as

$$F(q) = \int \mathrm{d}^4u \int_0^\infty \mathrm{d}\varkappa^2 \varepsilon(q_0 - u_0)\delta[(q-u)^2 - \varkappa^2]\varphi(u, \varkappa^2) . \tag{5}$$

$\varphi(u, \varkappa^2)$ is arbitrary if the vectors $(p+k)/2+u$ and $(p+k)/2-u$ both lie in the forward light-cone and

$$\varkappa \geqslant \mathrm{Max}\left\{0;\, m_1 - \sqrt{\left(\frac{p+k}{2}+u\right)^2};\, m_2 - \sqrt{\left(\frac{p+k}{2}-u\right)^2}\right\}.$$

φ vanishes outside this region. It depends, of course, also on the quantum numbers γ and on $p+k$. All our results will be based on applications of Dyson's theorem.

For the Fourier transform of the retarded commutator which appears in (2) or (3) we have the relation $(q' = (q_0', \boldsymbol{q}))$

$$F_R(q) = -\frac{1}{2\pi}\int \frac{\mathrm{d}q_0' F(q')}{q_0' - q_0}; \qquad \mathrm{Im}\, q_0 \geqslant 0 ,$$

if F_R is sufficiently bounded.

Inserting (5) gives

$$F_R(q) = -\frac{1}{2\pi}\int \mathrm{d}^4u \int \frac{\mathrm{d}\varkappa^2 \varphi(u, \varkappa^2)}{(q-u)^2 - \varkappa^2} . \tag{6}$$

In general we cannot expect F_R to be bounded enough for (6) to hold in this form. The necessary modification (⁸) does not alter the analytic properties we are interested in. It is therefore sufficient to discuss (6). We shall use it both for Eq. (2) and Eq. (3).

2. – Momentum transfer properties of scattering amplitudes.

By inserting (6) into Eq. (2) we obtain

$$T = \frac{1}{2\pi}\int \frac{\mathrm{d}^4u\, \mathrm{d}\varkappa^2 \varphi(u, \varkappa^2, p, k)}{((k'-p')/2 - u)^2 - \varkappa^2}, \tag{7}$$

(⁷) F. J. Dyson: *Phys. Rev.*, **110**, 1460 (1958).

(⁸) R. Jost and H. Lehmann: *Nuovo Cimento*, **5**, 1598 (1957), Eq. (4.5).

φ is an invariant function of the vectors u, p, k. The integration extends over the region given in (5). We choose the center of mass system to evaluate (7) and introduce the variables

$$(8)\qquad \begin{cases} W^2 = (p+k)^2; \qquad \Delta^2 = -\dfrac{(k-k')^2}{4}, \\ \text{or} \\ K^2 = \dfrac{[W^2-(m+\mu)^2][W^2-(m-\mu)^2]}{4W^2}; \qquad \cos\vartheta = 1 - \dfrac{2\Delta^2}{K^2}. \end{cases}$$

φ depends then only on u^2, u_0, $\boldsymbol{u}\cdot\boldsymbol{k}$, $\varkappa^2$, W. It vanishes outside

$$(9)\qquad \begin{cases} 0 \leqslant u \leqslant W/2; \quad -W/2+u \leqslant u_0 \leqslant W/2-u, \\ \varkappa \geqslant \operatorname{Max}\left\{0;\, m_1-\sqrt{(W/2+u_0)^2-u^2};\, m_2-\sqrt{(W/2-u_0)^2-u^2}\right\}. \end{cases}$$

Introducing polar co-ordinates in $\boldsymbol{u}$-space, (7) becomes:

$$(10)\qquad T(W,\cos\vartheta) = -\frac{1}{4\pi K}\int \mathrm{d}u_0 \int u\,\mathrm{d}u \int \mathrm{d}\varkappa^2 \int_0^{2\pi} \mathrm{d}\alpha \int_0^{\pi} \mathrm{d}\beta \cdot \frac{\varphi(u_0, u^2, \cos\alpha\sin\beta, \varkappa^2, W)}{\dfrac{K^2+u^2+\varkappa^2-(u_0+(m^2-\mu^2)/2W)^2}{2Ku\sin\beta} - \cos(\vartheta-\alpha)},$$

or

$$(11)\qquad T(W,\cos\vartheta) = \int_{x_0(W)}^{\infty} \mathrm{d}x \int_0^{2\pi} \mathrm{d}\alpha \, \frac{\overline{\varphi}(x, \cos\alpha, W)}{x-\cos(\vartheta-\alpha)},$$

with

$$\overline{\varphi}(x,\cos\alpha,W) = -\frac{1}{4\pi K}\int \mathrm{d}u_0 \int u\,\mathrm{d}u \int \mathrm{d}\varkappa^2 \int_0^{\pi} \mathrm{d}\beta \cdot \cdot\,\delta\left[x - \frac{K^2+u^2+\varkappa^2-(u_0+(m^2-\mu^2)/2W)^2}{2Ku\sin\beta}\right] \cdot \varphi(u_0, u^2, \cos\alpha\sin\beta, \varkappa^2, W).$$

The lower limit $x_0(W)$ is determined by

$$x_0(W) = \operatorname{Min}\left\{\frac{K^2+u^2+\varkappa^2-(u_0+(m^2-\mu^2)/2W)^2}{2Ku}\right\},$$

if u_0, u, $\varkappa$ vary over the region (9). The minimum can be calculated in an elementary manner. The result is

$$x_0(W) = \left[1 + \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{K^2[W^2 - (m_1 - m_2)^2]}\right]^{\frac{1}{2}} \tag{12}$$

In (11) the scattering angle $\cos\vartheta$ appears only in the denominator. We may therefore consider $\cos\vartheta$ as a complex variable and the scattering amplitude as an analytic function of $\cos\vartheta$. Moreover, this can be done separately for the real and imaginary parts of the amplitude.

Singularities of these functions can occur only if the denominator on the right hand side of (11) vanishes. That is for

$$\cos\vartheta = x\cdot\cos\alpha \pm i\sqrt{x^2 - 1}\,\sin\alpha\,.$$

We have therefore the following result:

(13) The real part and the imaginary part of the scattering amplitude are analytic functions of $\cos\vartheta$, regular inside an ellipse in the $\cos\vartheta$-plane with center at the origin and with axes x_0, $\sqrt{x_0^2 - 1}$.

We shall see presently—making use of Eq. (3)—that the imaginary part of the amplitude is regular in a larger domain, namely:

(14) $\operatorname{Im} T(W, \cos\vartheta)$ is regular in $\cos\vartheta$ inside an ellipse with center at the origin and with axes $2x_0^2 - 1$; $2x_0\cdot\sqrt{x_0^2 - 1}$. x_0 is given by (12).

Using $\cos\vartheta = 1 - (2\Delta^2/K^2)$ we can, of course, re-express (13) and (14) as analytic properties of the scattering amplitude as a function of momentum transfer.

These results (we defer the proof of (14)) lead—using well-known mathematical theorems (⁹)—to the following properties of the partial wave expansion of the scattering amplitude. Let

$$\left\{\begin{aligned} T(W, \cos\vartheta) &= \frac{1}{\pi^2}\frac{W}{K}\sum_{l=0}^{\infty}(2l+1)C_l(W)P_l(\cos\vartheta)\,,\\ \text{with}\quad C_l(W) &= \frac{\pi^2}{2}\frac{K}{W}\int_{-1}^{1} d\cos\vartheta\, T(W, \cos\vartheta)P_l(\cos\vartheta)\,. \end{aligned}\right. \tag{15}$$

(⁹) E. T. Whittaker and G. N. Watson. *A course of modern analysis*, 4th Ed., (Cambridge 1940), p. 322; G. Szegö, *Orthogonal Polynomials*, (New York, 1939), p. 238.

The Legendre series converges inside the domain of regularity of the represented functions; *i.e.* for $\cos\vartheta$ inside the ellipses (13) or (14) for $\operatorname{Re} T$ or $\operatorname{Im} T$ respectively. Also

$$\overline{\lim_{l\to\infty}}\,|\operatorname{Re} C_l(W)|^{1/l} \leqslant \frac{1}{x_0+\sqrt{x_0^2-1}}\,, \tag{13a}$$

$$\overline{\lim_{l\to\infty}}\,|\operatorname{Im} C\,(W)|^{1/l} \leqslant \frac{1}{(x_0+\sqrt{x_0^2-1})^2}\,. \tag{14a}$$

Taking into account the unitarity relation

$$\operatorname{Im} C_l(W) \geqslant [\operatorname{Re} C_l(W)]^2 + [\operatorname{Im} C_l(W)]^2\,, \tag{16}$$

we may note that (13*a*) is actually a consequence of (14*a*); *i.e.* if $\operatorname{Im} T(W, \cos\vartheta)$ is regular in the domain (14) it follows immediately that $\operatorname{Re} T(W, \cos\vartheta)$ is regular in (13).

We cannot conclude, of course, that the amplitude $T(W, \cos\vartheta)$ actually has singularities on the boundary of the domains (13) or (14). Using more physical information it may well be possible to improve these results.

To discuss the connection of the above statements with the non-physical region of dispersion relations, let us consider $\operatorname{Im} T(K^2, \Delta^2)$, the imaginary part of the amplitude, as a function of c.m. momentum and momentum transfer—the physical region is given by $K^2 > \Delta^2$. However, in the dispersion relation $\operatorname{Im} T(K^2, \Delta^2)$ is needed for all $K^2 > 0$. The expansion

$$\operatorname{Im} T(K^2, \Delta^2) = \frac{1}{\pi^2}\frac{W}{K}\sum_{l=0}^{\infty}(2l+1)\operatorname{Im} C_l(W)\, P_l\left(1-\frac{2\Delta^2}{K^2}\right), \tag{15a}$$

defines a continuation of $\operatorname{Im} T(K^2, \Delta^2)$ into the non-physical region. The series converges if

$$\Delta^2 < K^2x_0^2 = K^2 + \frac{(m_1^2-\mu^2)(m_2^2-m^2)}{W^2-(m_1-m_2)^2}\,;$$

i.e. it converges for all $K^2 > 0$ provided

$$\Delta^2 < \operatorname{Min}\{K^2x_0^2\}\,.$$

This leads to the restriction

$$\Delta^2 < \frac{8}{3}\frac{2m+\mu}{2m-\mu}\cdot\mu^2 \approx 3\mu^2\,,$$

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in the π-N case. ($\Delta^2 < 2\mu^2$ for equal particle scattering with $m_1 = m_2 = 2\mu$).

We have still to show that the continuation given by (15a) is indeed the correct definition of the non-physical region in the dispersion relation.

3. – Connection with dispersion relations.

To identify $\operatorname{Im} T(K^2, \Delta^2)$ as given by (15a) with the dispersion relation integrand it is convenient to consider (1) (which defines a function T for arbitrary real vectors k, k') not only on the energy shell ($k^2 = k'^2 = \mu^2$) but for the more general case $k^2 = k'^2 = \zeta$; keeping $p^2 = p'^2 = m^2$. T can then be considered as a function of

$$\omega = \frac{(k+k')(p+p')}{2\sqrt{(p+p')^2}} \qquad \zeta = k^2 = k'^2, \qquad \Delta^2 = -\frac{(p-p')^2}{4}.$$

We derive first Eq. (3).

(1) leads directly to

$$(17)\qquad \operatorname{Im} T(w, \zeta, \Delta^2) = \frac{1}{2}\int \mathrm{d}^4x \exp\left[i\left(\frac{k+k'}{2}\right)x\right]\langle p'|\left[j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right)\right]|p\rangle = \\ = \frac{1}{2}\left\{M(\omega, \zeta, \Delta^2) - M(-\omega, \zeta, \Delta^2)\right\},$$

with

$$(18)\qquad M(\omega, \zeta, \Delta^2) = \int \mathrm{d}^4x \exp\left[i\left(\frac{k+k'}{2}\right)x\right]\langle p'|j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right)|p\rangle = \\ = (2\pi)^4 \sum_\gamma \langle p'|j(0)|p+k, \gamma\rangle\langle p+k, \gamma|j(0)|p\rangle .$$

Let $\varphi\,\text{in}\,(p')$ denote the annihilation operator for an incoming nucleon with momentum p'. Then

$$\langle p'|j(0)|p+k, \gamma\rangle = \langle 0|\varphi\,\text{in}\,(p')j(0)|p+k, \gamma\rangle = \langle 0|[\varphi\,\text{in}\,(p'), j(0)]|p+k, \gamma\rangle$$
$$\text{if}\quad (p+k-p')^2 = k'^2 = \zeta < 9\mu^2$$

since $\langle 0|j(0)\varphi\,\text{in}\,(p')|p+k, \gamma\rangle = 0$ in this case. With the relation (⁵)

$$[\varphi\,\text{in}\,(p'), j(0)] = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^4x \exp[ip'x]R'A(0)\psi(x) ,$$

and an analogous treatment of the second factor in (18) we have

$$(19)\quad M(\omega, \zeta, \Delta^2) = 2\pi \int d^4x_1 d^4x_2 \exp\,[ip'x_1 - ipx_2] \sum_{\gamma} \langle 0|R'A(0)\psi(x_1)|p+k, \gamma\rangle \cdot$$

$$\cdot \langle p+k, \gamma|R'A(0)\psi^{+}(x_2)|0\rangle = 2\pi \int d^4x_1 d^4x_2 \exp\left[i\left(\frac{k'-p'}{2}\right)x_1 - i\left(\frac{k-p}{2}\right)x_2\right] \cdot$$

$$\cdot \sum_{\gamma} \langle 0|R'A\left(\frac{x_1}{2}\right)\psi\left(-\frac{x_1}{2}\right)|p+k, \gamma\rangle \cdot \langle p+k, \gamma|R'A\left(\frac{x_2}{2}\right)\psi^{+}\left(-\frac{x_2}{2}\right)|0\rangle\,.$$

We note that the imaginary part of the physical scattering amplitude is given by

$$(20)\qquad \operatorname{Im} T = \tfrac{1}{2} M(\omega, \mu^2, \Delta^2)\,.$$

The term $M(-\omega, \mu^2, \Delta^2)$ does not contribute since it vanishes for $\omega \geqslant \sqrt{\Delta^2+\mu^2}$. Hence (19) is on the energy shell equivalent to (3).

To find analytic properties of M as given by (19) we use again the integral representation (6). This leads to

$$(21)\quad M(\omega, \zeta, \Delta^2) = \frac{1}{2\pi} \int \frac{d^4u_1\, d\varkappa_1^2\, d^4u_2\, d\varkappa_2^2\, \Phi(u_1, \varkappa_1, u_2, \varkappa_2, p+k)}{[((k'-p')/2 - u_1)^2 - \varkappa_1^2][((k-p)/2 - u_2)^2 - \varkappa_2^2]}\,,$$

where

$$\Phi(u_1, \varkappa_1, u_2, \varkappa_2, p+k) = \sum_{\gamma} \varphi_{\gamma}(u_1, \varkappa_1, p+k)\, \varphi_{\gamma}^{*}(u_2, \varkappa_2, p+k)$$

is built up from the weight functions φ_{γ} corresponding to the individual terms on the right hand side of (19). Φ is a real, invariant function which satisfies the support conditions (6) in each pair of variables u, $\varkappa$ separately. We choose the center of mass system and replace ω by

$$W^2 = 2\omega\sqrt{\Delta^2 + m^2} + 2\Delta^2 + m^2 + \zeta\,.$$

Then

$$(22)\qquad M(\omega, \zeta, \Delta^2) = \frac{1}{2\pi} \cdot$$

$$\cdot \int \frac{d^4u_1\, d\varkappa_1^2\, d^4u_2\, d\varkappa_2^2\, \Phi(u_{10}, u_1^2, \varkappa_1^2, u_{20}, u_2^2, \varkappa_2^2, \mathbf{u}_1\mathbf{u}_2/u_1u_2, W)}{[((m^2-\zeta)/2W + u_{10})^2 - \varkappa_1^2 - (\mathbf{k}'-\mathbf{u}_1)^2][((m^2-\zeta)/2W + u_{20})^2 - \varkappa_2^2 - (\mathbf{k}-\mathbf{u}_2)^2]}\,.$$

With polar co-ordinates

$$(23)\qquad M(W, \zeta, \Delta^2)\, \frac{1}{8\pi\, K^2(\zeta)} \int du_0\, u_i\, du_i\, d\varkappa_i^2 \int_0^{2\pi} d\alpha \int_0^{\pi} d\beta_1 \int_0^{\pi} d\beta_2 \int_0^{2\pi} d\chi \cdot$$

$$\cdot \frac{\Phi(u_0, u_i^2, \varkappa_i^2, \cos\alpha \sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2, W)}{[x_1(\zeta) - \cos(\vartheta - \chi)][x_2(\zeta) - \cos(\chi - \alpha)]}\,,$$

$$x_i(\zeta) = \frac{K^2(\zeta) + u_i^2 + \varkappa_i^2 - ((m^2 - \zeta)/2W + u_{i0})^2}{2K(\zeta)u_i \sin \beta_i};$$

$$K^2(\zeta) = \frac{(W^2 + m^2 - \zeta)^2 - 4m^2W^2}{4W^2}.$$

We note that

$$\int_0^{2\pi} \mathrm{d}\chi \, \frac{1}{[x_1 - \cos(\vartheta - \chi)]} \cdot \frac{1}{[x_2 - \cos(\chi - \alpha)]} =$$

$$= 2\pi \, \frac{(x_1/(\sqrt{x_1^2 - 1})) + (x_2/(\sqrt{x_2^2 - 1})}{(x_1x_2 + \sqrt{x_1^2 - 1}\sqrt{x_2^2 - 1} - \cos(\vartheta - \alpha)}.$$

From (23) analytic properties in ζ and Δ^2 follow. On the energy shell, *i.e.* $\zeta = \mu^2$, we can introduce $y = x_1x_2 + \sqrt{x_1^2 - 1}\sqrt{x_2^2 - 1}$ as a new integration variable. Only integrations over y and α remain; the other integrations result only in a new weight function $\bar{\Phi}(y, \cos\alpha, W)$.

The minimum value of y is

$$y_0 = \mathrm{Min}\left\{x_1x_2 + \sqrt{x_1^2 - 1}\sqrt{x_2^2 - 1}\right\} = 2x_0^2 - 1\,.$$

Therefore

$$M(W^2, \Delta^2) = 2\,\mathrm{Im}\,T = \int_{2x_0^2-1}^{\infty} \mathrm{d}y \int_0^{2\pi} \mathrm{d}\alpha \, \frac{\bar{\Phi}(y, \cos a, W)}{y - \cos(\vartheta - \alpha)}\,.$$

This proves the statement (14).

It can be seen now that the analytic continuation of $\mathrm{Im}\,T$ defined by (15*a*) yields the correct non-physical part of the dispersion relation integrand. In the proofs of these relations (3,4) it is shown first that—as a consequence of Eq. (1)—a dispersion relation in ω holds if ζ is taken real and $\zeta < -\Delta^2$. The absorptive part in this relation is $M(W, \zeta, \Delta^2)$ as given by (18) and (23). The dispersion relation for the physical value $\zeta = \mu^2$ is then obtained by analytic continuation in ζ, provided $M(W, \zeta, \Delta^2)$ is an analytic function of ζ regular for $\mathrm{Re}\,\zeta \leqslant \mu^2$ in a neighborhood of the real axis. It follows from (23) that this condition is satisfied if

$$\Delta^2 < \mathrm{Min}\,\{K^2 x_0^2\}\,.$$

This is also the condition for the convergence of the Legendre series. The absorptive part of the dispersion relation is then given by (23) with $\zeta = \mu^2$; *i.e.* the non-physical region is obtained by analytic continuation in Δ^2 which can be carried out by the Legendre expression (15*a*).

The possibility of evaluating the non-physical region in this manner has been discussed earlier ([5]). While no proofs were given, it was believed on the basis of threshold arguments that such a procedure could be valid only if $\Delta^2 < \mu^2$, due to a branch point of the scattering amplitude as a function of Δ^2. We have shown that the expansion converges also for higher values of Δ^2; the limit being $\Delta^2 = 2\mu^2$ in the case of equal particle scattering. We believe that this is due to the fact that the real and imaginary part of the amplitude are separately analytic functions of Δ^2 and have different properties. For the dispersion relation only the imaginary part is needed and it has a larger domain of regularity. The mentioned branch point is likely to be present in the real part.

While we have no good reason to believe that our results are best possible, the expected appearance of a singularity in the real part gives us—via the unitarity relation (16)—an upper limit to the values of Δ^2 for which the Legendre expansion for the imaginary part might converge. In the case of equal particle scattering the expected branch point of the real part at $\Delta^2 = W^2/4$ leads to the limitation $\Delta^2 < 8\mu^2$ for the Legendre expansion of the imaginary part.

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Determination of the Pion-Nucleon Scattering Amplitude from Dispersion Relations and Unitarity. General Theory

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A method is proposed for using relativistic dispersion relations, together with unitarity, to determine the pion-nucleon scattering amplitude. The usual dispersion relations by themselves are not sufficient, and we have to assume a representation which exhibits the analytic properties of the scattering amplitude as a function of the energy and the momentum transfer. Unitarity conditions for the two reactions $\pi+N\rightarrow\pi+N$ and $N+\bar{N}\rightarrow 2\pi$ will be required, and they will be approximated by neglecting states with more than two particles. The method makes use of an iteration procedure analogous to that used by Chew and Low for the corresponding problem in the static theory. One has to introduce two coupling constants; the pion-pion coupling constant can be found by fitting the sum of the threshold scattering lengths with experiment. It is hoped that this method avoids some of the formal difficulties of the Tamm-Dancoff and Bethe-Salpeter methods and, in particular, the existence of ghost states. The assumptions introduced are justified in perturbation theory.

As an incidental result, we find the precise limits of the region for which the absorptive part of the scattering amplitude is an analytic function of the momentum transfer, and hence the boundaries of the region in which the partial-wave expansion is valid.

1. INTRODUCTION

In recent years dispersion relations have been used to an increasing extent in pion physics for phenomenological and semiphenomenological analyses of experimental data,[1] and even for the calculation of certain quantities in terms of the pion-nucleon scattering amplitude.[2] It is therefore tempting to ask the question whether or not the dispersion relations can actually replace the more usual equations of field theory and be used to calculate all observable quantities in terms of a finite number of coupling constants—a suggestion first made by Gell-Mann.[3] At first sight, this would appear to be unreasonable, since, although it is necessary to use all the general principles of quantum field theory to derive the dispersion relations, one does not make any assumption about the form of the Hamiltonian other than that it be local and Lorentz-invariant. However, in a perturbation expansion these requirements are sufficient to specify the Hamiltonian to within a small number of coupling constants if one demands that the theory be renormalizable and therefore self-consistent. It is thus very possible that, even without a perturbation expansion, these requirements are sufficient to determine the theory. In fact, if the "absorptive part" of the scattering amplitude, which appears under the integral sign of the dispersion relations, is expressed in terms of the scattering amplitude by means of the unitarity condition, one obtains equations which are very similar to the Chew-Low[4] equations in static theory. These equations have been used by Salzman and Salzman[5] to obtain the pion-nucleon scattering phase shifts.

It is the object of this paper to find a relativistic analog of the Chew-Low-Salzman method, which could be used to calculate the pion-nucleon scattering amplitude in terms of two coupling constants only. As in the static theory, the unitarity equation will involve the transition amplitude for the production of an arbitrary number of mesons, and, in this case, of nucleon pairs as well. In order to make the equations manageable, it is necessary to neglect all but a finite number of processes; as a first approximation, the "one-meson" approximation, we shall neglect all processes except elastic scattering.

The equations obtained from the dispersion relations and the one-meson approximation differ from the static Chew-Low equations in two important respects. Whereas, in the static theory, there was only P-wave scattering, we now have an infinite number of angular momentum states, and the crossing relation, if expressed in terms of angular momentum states, would not converge. Further, in the relativistic theory, the dispersion relations involve the scattering amplitude in the "unphysical" region, i.e., through angles whose cosine is less than -1. For these reasons, the method of procedure will be more involved than in the static theory. We shall require, not only the analytic properties of the scattering amplitude as a function of energy for fixed momentum transfer, which are expressed by the dispersion relations, but its analytic properties as a function of both variables. The required analytic properties have not yet been proved to be consequences of microscopic causality. In order to carry out the proof, one would almost certainly have to consider simultaneously several Green's functions together with the equations connecting them which follow from unitarity. It is unlikely that such a program will be carried through in the immediate future. However, if the solution obtained by the use of these analytic properties were to be expanded in a perturbation series, we would obtain precisely those terms of the usual perturbation series included in the one-meson approximation. The assumed analytic properties are, therefore, probably correct, at any rate in the one-meson approximation.

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[3] M. Gell-Mann, *Proceedings of the Sixth Annual Rochester Conference High-Energy Physics, 1956* (Interscience Publishers, Inc., New York, 1956), Sec. III, p. 30.

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As we have to resort to perturbation theory in order to justify our assumptions, we do not yet have a theory in which the general principles of quantum theory are supplemented only by the assumption of microscopic causality. Nevertheless, the approximation scheme used has several advantages over the approximations previously applied to this problem, such as the Tamm-Dancoff or Bethe-Salpeter approximations. It refers throughout only to renormalized masses and coupling constants. The Tamm-Dancoff equations, by contrast, are unrenormalizable in higher approximations and the Bethe-Salpeter equations, while they are covariant and therefore renormalizable in all approximations, present difficulties of principle when one attempts to solve them. Further, we may hope that the one-meson approximation is more accurate than the Tamm-Dancoff approximations. The latter assumes that those components of the state vector containing more than a certain number of bare mesons are negligibly small—an approximation that is known to be completely false for the experimental value of the coupling constant. The one-meson approximation, on the other hand, assumes that the cross section for the production of one or more *real* mesons is small except at high energies. While this approximation is certainly not quantitatively correct, it is nevertheless probably a good deal more accurate than the Tamm-Dancoff approximation. Finally, the one-meson approximation, unlike the Tamm-Dancoff or Bethe-Salpeter approximations, possesses crossing symmetry. Now it is very probable that the "ghost states" which have been plaguing previous solutions of the field equations are due to the neglect of crossing symmetry. As evidence of this, we may cite the case of charged scalar theory without recoil, for which the one-meson approximation has been solved completely.[6,7] The solution obtained with neglect of the crossing term possesses the usual ghost state if the source radius is sufficiently small. The Lee model,[8] which has no crossing symmetry, shows a similar behavior. If the crossing term in the charged scalar model is included, however, there is no ghost state.

It has been pointed out by Castillejo, Dalitz, and Dyson[7] that the dispersion relations, at any rate in the charged scalar model, do not possess a unique solution. This might have been expected, since it is possible to alter the Hamiltonian without changing the dispersion relations. One simply has to introduce into the theory a baryon whose mass is greater than the sum of the masses of the meson and nucleon. Such a baryon would be unstable, and would therefore not appear as a separate particle or contribute a term to the dispersion relations. In perturbation theory, the simplest of the solutions found by Castillejo, Dalitz, and Dyson, i.e., the solution without any zero in the scattering amplitude, agrees with the solution obtained from a Hamiltonian in which there are no unstable particles, and the more complicated solutions correspond to the existence of unstable baryons. We shall assume that this is so independently of perturbation theory, and shall concern ourselves with the simplest solution. There is no physical reason why one of the other solutions may not be the correct one, but it seems worthwhile to try to compare with experiment the consequences of a theory without unstable particles. It should in any case be emphasized that the ambiguity is not a specific feature of this method of solution, but is inherent in the theory itself. The difference is that, in other methods, it occurs in writing down the equations, whereas in this method it occurs in solving them.

In Sec. 2 we shall discuss the analytic properties of the scattering amplitude, and, in Sec. 3, we shall show how these properties can be used together with the unitarity condition to solve the problem. We shall in this section ignore the "subtraction terms" in the dispersion relations. As in the corresponding static problem, we have to use an iteration procedure in which the crossing term is taken from the result of the previous iteration. The details of this solution will be entirely different from the static problem, the reason being that the part of the amplitude corresponding to the lowest angular momentum states, which is a polynomial in the momentum transfer, actually appears as a subtraction term in the dispersion relation with respect to this variable and has thus not yet been taken into account. In this and the next section we shall also be able to specify details of the analytic representation that were left undetermined in Sec. 2, in particular, we shall be able to give precise limits to the values of the momentum transfer within which the partial-wave expansion converges. In Sec. 4 we shall investigate the subtraction terms in the dispersion relations. We shall find that, in order to determine them, we shall require the unitarity condition for the lowest angular momentum states, not only in pion-nucleon scattering, but also in the pair-annihilation reaction $N+\bar{N}\rightarrow 2\pi$, which is represented by the same Green's function. The coupling constant for meson-meson scattering is thus introduced into the theory; as its value is not known experimentally it will have to be determined by fitting one of the results of the calculation, such as the sum of the S-wave scattering lengths at threshold, with experiment. The calculations

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[7] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956).
[8] K. W. Ford, Phys. Rev. **105**, 320 (1957).

of these low angular momentum states would be done in the same spirit as the Chew-Low calculations, and the details will not be given in this paper. We thus have a procedure in which the first few angular momentum states are calculated by methods similar to those used in the static theory, while the remaining part of the scattering amplitude, which will be called the "residual part," is calculated by a different procedure which does not make use of a partial-wave expansion. Needless to say, the two parts of the calculation become intermingled by the iteration procedure.

It is only in the calculation of the subtraction terms that use has to be made of the unitarity condition for the pair-annihilation reaction. For the residual part, it is only necessary to use the unitarity condition for pion-nucleon scattering. Had it been possible to use the unitarity condition exactly instead of in the one-meson approximation, the result would also satisfy the unitarity condition for the annihilation reaction in a consistent theory. As it is, we find that the residual part consists of a number of terms which correspond to various intermediate states in the annihilation reaction. In Sec. 5 it is pointed out that the calculation is greatly simplified if we keep only those terms of the residual part corresponding to pair annihilation through states with fewer than a certain number of particles. Such an approximation has already been made in calculating the subtraction terms. The unitarity condition for pion-nucleon scattering is no longer satisfied except for the low angular momentum states. However, the terms neglected are of the order of magnitude of, and probably less than, terms already neglected. The two reactions of pion-nucleon scattering and pair annihilation are now treated on an equivalent footing.

It will be found that the unitarity condition, in the one-meson approximation, cannot be satisfied at all energies if crossing symmetry and the analytic properties are to be maintained. The reason is that the unitarity condition for the scattering reaction is not completely independent of the unitarity condition for the "crossed" reaction with the two pions interchanged, and they contradict one another if an approximation is made. There is, of course, no difficulty in the region where the one-meson approximation is exact. For sufficiently small values of the coupling constant, we shall still be able to obtain a unique procedure. For values of the coupling constant actually encountered, one part of the crossing term may have to be cut off at the threshold for pair production in pion-nucleon scattering. It is unlikely that the result will be sensitive to the form and the precise value of the cutoff.

2. DISPERSION RELATIONS AND ANALYTICITY PROPERTIES OF THE TRANSITION AMPLITUDE

The kinematical notation to be used in writing down the dispersion relations will be similar to that of Chew *et al.*[1] The momenta of the incoming and outgoing pions will be denoted by q_1 and q_2, those of the incoming and outgoing nucleons by p_1 and p_2. We can then define two invariant scalars

$$\nu = -(p_1+p_2)(q_1+q_2)/4M, \tag{2.1}$$

$$t = -(q_1-q_2)^2. \tag{2.2}$$

The latter is minus the square of the invariant momentum transfer. The laboratory energy will be given by the equation

$$\omega = \nu - (t/4M). \tag{2.3a}$$

It is more convenient to use, instead of the laboratory energy, the square of the center-of-mass energy (including both rest-masses), which is linearly related to it by the equation

$$s = M^2+\mu^2+2M\omega. \tag{2.3b}$$

The Green's function relevant to the process under consideration,

$$\pi_1+N_1 \rightarrow \pi_2+N_2, \tag{I}$$

also gives the processes

$$\pi_2+N_1 \rightarrow \pi_1+N_2 \tag{II}$$

and

$$N_1+\bar{N}_2 \rightarrow \pi_1+\pi_2. \tag{III}$$

The matrix elements for the process II can be obtained from those for the process I by crossing symmetry; the laboratory energy and the square of the center-of-mass energy will now be

$$\omega_c = -\nu - (t/4M) = -\omega - (t/2M), \tag{2.4a}$$

$$s_c = M^2+\mu^2+2M\omega_c = 2M^2+2\mu^2-s-t. \tag{2.4b}$$

The square of the momentum transfer will be $-t$ as before. For the process III, the square of the center-of-mass energy will be t. The square of the momentum transfer between the nucleon N_1 and the pion π_2 will be s_c and that between the nucleon N_1 and the pion π_1 will be s.

The kinematics for the three reactions are represented diagrammatically in Fig. 1 in which t has been plotted against ν. AB represents the line $s=(M+\mu)^2$, or $\omega=\mu$, and lines for which s is constant will be parallel to it. The region for which the process I is energetically possible is therefore that to the right of AB. However, only the shaded part of this area is the "physical region"; in the unshaded part, though the energy of the meson is greater than its rest-mass, the cosine of the scattering angle is not between -1 and $+1$. The physical region is bounded above by the line $t=0$, i.e., the line of forward scattering, and below by the line of backward scattering. Similarly CD is the line $s_c=(M+\mu)^2$; the region for which the process II is energetically possible is that to the left of CD, and the shaded area represents the physical region for this reaction. Lines of constant energy for the reaction III are horizontal lines. The reaction will be energetically possible above the line EF, at which $t=4M^2$, and again the shaded area represents the physical region.

We now examine the analytic properties of the scattering amplitude. To simplify the writing, we shall first neglect spin and isotopic spin; the transition amplitude will then be a scalar function $A(\nu,t)$ of the two invariants ν and t. Its analytic properties as a function of ν, with t constant, are exhibited by the usual dispersion relations

$$A(\nu,t)=\frac{g^2}{2M}\left(\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right)+\frac{1}{\pi}\int_{\mu+(t/4M)}^{\infty}d\nu'\,\frac{A_1(\nu',t)}{\nu'-\nu}$$

$$-\frac{1}{\pi}\int_{-\infty}^{-\mu-(t/4M)}d\nu'\,\frac{A_2(\nu',t)}{\nu'-\nu},\quad (2.5)$$

where $\nu_B=-(\mu^2/2M)+(t/4M)$. In this and all subsequent such equations, the energy denominators are taken to have a small imaginary part. A_1 and A_2 are the "absorptive parts" associated with the reactions I and II, respectively, and are given by the equations

$$(2\pi)^4A_1(\nu,t)\delta(p_1+q_1-p_2-q_2)=(2\pi)^6\left(\frac{4p_{01}p_{02}q_{01}q_{02}}{M^2}\right)^{\frac{1}{2}}$$

$$\times\sum_n\langle N(p_1)\pi(q_1)|n\rangle\langle n|N(p_2)\pi(q_2)\rangle,\quad (2.6)$$

$$(2\pi)^4A_2(\nu,t)\delta(p_1+q_1-p_2-q_2)=(2\pi)^6\left(\frac{4p_{01}p_{02}q_{01}q_{02}}{M^2}\right)^{\frac{1}{2}}$$

$$\times\sum_n\langle N(p_1)\pi(-q_2)|n\rangle\langle n|N(p_2)\pi(-q_1)\rangle.\quad (2.7)$$

The symbol $\langle N(p_1)\pi(q_1)|$ denotes a state with an ingoing nucleon of momentum p_1 and an ingoing pion of momentum q_1. The sum $\sum_n$ is to be taken over all intermediate states. A_1 and A_2 are nonzero to the right of AB, and to the left of CD, respectively.

Equation (2.5) indicates that A is an analytic function of ν in the complex plane, with poles at $\pm\nu_B$, and cuts along the real axis from $\mu+(t/4M)$ to ∞ and from $-\infty$ to $-\mu-(t/4M)$.

On Fig. 1, (2.5) will be represented by an integration along a horizontal line below the ν axis. The poles will occur where this line crosses the dashed lines; apart from them, the integrand will be zero between AB and CD. Except for forward scattering, the region where the integrand is nonzero will lie partly in the unphysical region, where the energy is above threshold but the angle imaginary.

Equation (2.5) is only true as it stands if the functions A, A_1, and A_2 tend to zero sufficiently rapidly as ν tends to infinity; otherwise it will be necessary to perform one or more subtractions in the usual way. Whenever such a dispersion relation is written down, the possibility of having to perform subtractions is implied.

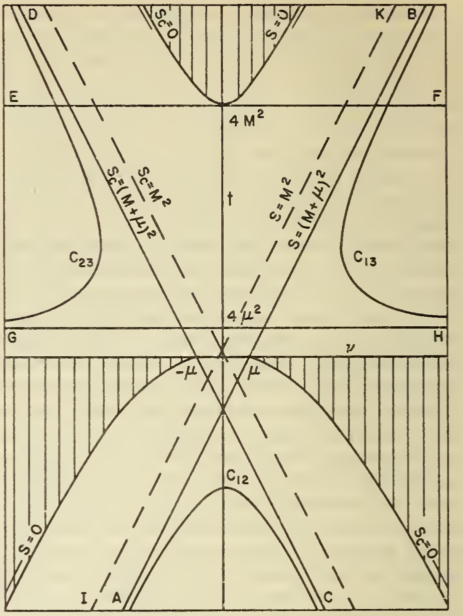

FIG. 1. Kinematics of the reactions I, II, and III.

We next wish to obtain analytic properties of A as a function of t. In order to do this we shall write the scattering amplitude, not as the expectation value of the time-ordered product of the two meson current operators between two one-nucleon states, as is done in the proof of the usual dispersion relations,[9,10] but as the expectation value of the product of a meson current operator and a nucleon current operator between a nucleon state and a meson state. Thus

$$(2\pi)^4A\delta(p_1+p_2-q_1-q_2)=(2\pi)^3\left(\frac{2p_{01}q_{02}}{M}\right)^{\frac{1}{2}}i\int dxdx'$$

$$\times e^{-iq_1x+ip_2x'}\langle N(p_1)|T\{j(x)\bar{a}(x')\}|\pi(q_2)\rangle,\quad (2.8)$$

where $a(x')$ is a nucleon current operator. From this expression, we can obtain dispersion relations in which the momentum transfer between the incoming nucleon and the outgoing pion, rather than between the two nucleons, is kept constant—the proof is exactly the same as the usual heuristic proof of the ordinary dispersion relations.[9,10] As this momentum transfer is just s_c, we obtain dispersion relations in which s_c is kept constant; if A is written as a function of s_c and t, they

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[9] M. L. Goldberger, Phys. Rev. 99, 979 (1955).

[10] R. H. Capps and G. Takeda, Phys. Rev. 106, 1337 (1956).

take the form

$$A(s_c,t)=\frac{g^2}{s_c+t-M^2-2\mu^2}-\frac{1}{\pi}\int_{\infty}^{(M-\mu)^2-s_c}dt'\,\frac{A_1(s_c,t')}{t'-t}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\,\frac{A_3(s_c,t')}{t'-t}.\qquad(2.9)$$

The absorptive parts in the integrand are as usual obtained by replacing the time-ordered product in (2.8) by half the commutator. The first term, in which the operators are in the order $j(x)\bar{a}(x')$, is exactly A_1, and will therefore be nonzero to the right of AB and have a δ function along IK. The second term, however, in which the operators are in the order $\bar{a}(x')j(x)$, will now be related to the process III. It will be given by the equation

$$(2\pi)^4A_3(s_c,t)\delta(p_1+q_1-p_2-q_2)=(2\pi)^6\left(\frac{4p_{01}p_{02}q_{01}q_{02}}{M^2}\right)^{\frac{1}{2}}\times\sum_n\langle N(p_1)\bar{N}(-p_2)|n\rangle\langle n|\pi(-q_1)\pi(q_2)\rangle.\qquad(2.10)$$

The state n of lowest energy will now be the two-meson state. A_3 will therefore be nonzero above the line $t=4\mu^2$, represented by GH in Fig. 1 (since t is square of the center-of-mass energy of the process III). The dispersion relation (2.10) is represented by an integration along a line parallel to CD and to the right of the line $s_c=0$. It implies that A is an analytic function of t for fixed s_c, with a pole at $t=M^2+2\mu^2-s_c$, and cuts along the real axis from $-\infty$ to $(M-\mu)^2-s_c$ and from $4\mu^2$ to ∞.

As in the usual dispersion relation, part of the range of integration in Eq. (2.9) will lie in the unphysical region. This region now includes, besides imaginary angles at permissible energies, the entire area between the lines $t=4\mu^2$ and $t=4M^2$, where there are contributions to A_3 from intermediate states with two or more pions. The rigorous proof of (2.9) is therefore much more difficult than that of (2.5), and probably cannot be carried out without introducing the unitarity equations.

By interchanging the two pions in the expression (2.8), we can obtain a third dispersion relation in which s is kept constant:

$$A(s,t)=\frac{g^2}{s+t-M^2-2\mu^2}-\frac{1}{\pi}\int_{-\infty}^{(M-\mu)^2-s}dt'\,\frac{A_2(s,t')}{t'-t}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\,\frac{A_3(s,t')}{t'-t}.\qquad(2.11)$$

On Fig. 1, this would be represented by an integration along a line parallel to AB, and to the left of the line $s=0$.

Let us now try to obtain the analytic properties of A considered as a function of two complex variables. The simplest assumption we could make is that it is analytic in the entire space of the two variables except for cuts along certain hyperplanes. We can then determine the location of the cuts from the requirement that A must satisfy the dispersion relations (2.5), (2.9), and (2.11); there will be a cut when s is real and greater than $(M+\mu)^2$, a cut when s_c is real and greater than $(M+\mu)$, and a cut when t is real and greater than $4\mu^2$. The discontinuities across these cuts will be, respectively, $2A_1$, $2A_2$, and $2A_3$. In addition, A will have poles when $s=M^2$ and when $s_c=M^2$. By a double application of Cauchy's theorem, it can be shown that a function with cuts and poles in these positions can be represented in the form

$$A=\frac{g^2}{M^2-s}+\frac{g^2}{M^2-s_c}+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{4\mu^2}^{\infty}dt'\,\frac{A_{13}(s',t')}{(s'-s)(t'-t)}+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds_c'\int_{4\mu^2}^{\infty}dt'\,\frac{A_{23}(s_c',t')}{(s_c'-s_c)(t'-t)}+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{(M+\mu)^2}^{\infty}ds_c'\,\frac{A_{12}(s',s_c')}{(s'-s)(s_c'-s_c)}.\qquad(2.12)$$

This is a generalization of a representation first suggested by Nambu.[11] While we have for convenience used the three variables s, s_c, and t, which are the energies of the three processes, they are connected by the relation

$$s+s_c+t=2(M^2+\mu^2),\qquad(2.13)$$

so that A is really a function of two variables only. A_{13}, A_{23} and A_{12}, which will be referred to as the "spectral functions," are nonzero in the regions indicated at the top right, top left and bottom of Fig. 1. The precise boundaries C_{13}, C_{23}, and C_{12} of the regions will be determined by unitarity in the following sections; from the reasoning given up till now, all that can be said is that the regions must lie within the respective triangles as indicated, and that the boundary must approach the sides of the triangles asymptotically (or it could touch them at some finite point). The spectral functions are always zero in the physical region.

As in the case of ordinary dispersion relations, the representation (2.12) will not be true as it stands, but will require subtractions. The subtractions will modify one or both of the energy denominators in the usual way and, in addition, they will require the addition of extra terms. These terms will not now be constants, but functions of one of the variables, e.g., if there is a subtraction in the s integration of the first term, the extra term will be a function of t. These functions must then have the necessary analytic properties in their variables, so that they will have the form

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[11] Y. Nambu, Phys. Rev. 100, 394 (1955).

$$\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds'\,\frac{f_1(s')}{s'-s}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} ds_c'\,\frac{f_2(s_c')}{s_c'-s_c}+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\,\frac{f_3(t')}{t'-t}. \quad (2.14)$$

If more than one subtraction is involved, we may have similar terms multiplied by polynomials. Even if the spectral functions in (2.12) tend to zero as one of the variables tends to infinity, so that no subtraction in that variable is necessary, it is still not precluded that the corresponding term in (2.14) does not appear, as the function still has the required analytic properties. For pion-nucleon scattering, however, there is no undetermined over-all term, independent of both variables, to be added, as the requirement that the scattering amplitude for each angular momentum wave have the form $e^{i\delta}\sin\delta/k$, with $\mathrm{Im}\delta<0$, forces A to tend to zero in the physical region when both s and t become infinite.

The Nambu representations for the complete Green's functions are known to be invalid, even in the lowest nontrivial order of perturbation theory. The representation quoted here, however, restricts itself to the mass shells of the particles, and has not been shown to be invalid. In fact, in the case of Compton scattering, the fourth-order terms, which have been worked out by Brown and Feynman,[12] are found to have this representation, and, as we have stated in the introduction, all the perturbation terms included in the one-meson approximation can be similarly represented.

The dispersion relations are an immediate consequence of the representation (2.12). To obtain the usual dispersion relation (2.5), the third integral in (2.12) must be written as[13]

$$-\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds'\int_{-\infty}^{t_2(s)} dt'\,\frac{A_{12}(s',t')}{(s'-s)(t'-t)}-\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds'\int_{-\infty}^{t_2(s_c)} dt'\,\frac{A_{12}(s_c',t')}{(s_c'-s_c)(t'-t)}.$$

It then follows that

$$A=\frac{g^2}{M^2-s}+\frac{g^2}{M^2-s_c}+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds'\,\frac{A_1(s',t)}{s'-s}+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty} ds_c'\,\frac{A_2(s_c',t)}{s_c'-s_c}, \quad (2.15)$$

where

$$A_1(s,t)=\frac{1}{\pi}\int_{t_1(s)}^{\infty} dt'\,\frac{A_{13}(s,t')}{t'-t}-\frac{1}{\pi}\int_{-\infty}^{t_2(s)} dt'\,\frac{A_{12}(s,t')}{t'-t}, \quad (2.16)$$

$$A_2(s_c,t)=\frac{1}{\pi}\int_{t_1(s_c)}^{\infty} dt'\,\frac{A_{23}(s_c,t')}{t'-t}-\frac{1}{\pi}\int_{-\infty}^{t_2(s_c)} dt'\,\frac{A_{12}(s_c,t')}{t'-t}. \quad (2.17)$$

Equation (2.15) is, however, just the dispersion relation (2.5), since s, s_c, and ν are connected by the relations (2.4) and t is being kept constant. We also see that the absorptive parts A_1 and A_2 themselves satisfy dispersion relations in t, with s (or s_c) constant; the imaginary parts which appear in the integrand are now simply the spectral functions. Equation (2.16) will be represented in Fig. 1, by an integration along a line parallel to AB and to the right of it. The limits t_1 and t_2 are the points at which this line crosses the curves C_{13} and C_{12}. They satisfy the inequalities

$$t_1>4\mu^2, \quad (2.18a)$$

$$t_2<(M-\mu)^2-s. \quad (2.18b)$$

A_1 will be nonzero for $s>(M+\mu)^2$, as it should, as long as the curves C_{13} and C_{12} approach the line AB at some point and do not cross it.

The dispersion relations (2.9) and (2.11) can be proved from (2.12) in a similar way; the absorptive part A_3 will then satisfy a dispersion relation in ν with s constant:

$$A_3=\frac{1}{\pi}\int_{\nu_3(t)}^{\infty} d\nu'\,\frac{A_{13}(\nu',t)}{\nu'-\nu}-\frac{1}{\pi}\int_{-\infty}^{-\nu_3(t)} d\nu'\,\frac{A_{23}(\nu',t)}{\nu'-\nu}. \quad (2.19)$$

This dispersion relation will be represented by an integration along a horizontal line above GH. ν_3 and $-\nu_3$ will be the points at which the line of integration crosses C_{13} and C_{23}.

Finally, then, the scattering amplitude A satisfies dispersion relations in which any of the quantities t, s_c, and s are kept constant. Further, it follows from (2.12), by the reasoning just given, that the values of the quantity which is being kept constant need no longer be restricted in sign. Thus, for example, we now know the analytic properties of A, as a function of momentum transfer, for fixed energy greater than (as well as less than) $(M+\mu)^2$. They are given by the dispersion relation (2.11), so that A is an analytic function of the square of the momentum transfer, with a pole at $t=M^2+2\mu^2-s$, and cuts along the real axis from $t=4\mu^2$ to ∞ and from $t=-\infty$ to $(M-\mu)^2-s$. For $s>(M+\mu)^2$, these cuts and poles are entirely in the nonphysical region. It has already been shown rigorously by Lehmann[14] that A is analytic in t in an area including the physical region. The absorptive parts A_1, A_2 and A_3 will themselves satisfy dispersion relations, provided that the correct variable be kept constant (s, s_c, and t for A_1, A_2, and A_3, respectively). The weight functions for these dispersion relations are entirely in the nonphysical region, and the boundaries of the areas in which they are nonzero are yet to be determined. In particular, we see that the absorptive part A_1 has the same analytic properties as a function of the momentum transfer [for s constant and greater than $(M+\mu)^2$] as the scattering amplitude, except that there is now no pole, and the cuts only extend from t_1 to ∞ and from $-\infty$ to t_2. According to the inequalities (2.14), these cuts do not reach as far inward as the cuts of A considered as a function of the momentum transfer. This agrees with another result of Lehmann[14] who showed that the region of analyticity of A_1 as a function of t was larger than the region of analyticity of A as a function of t.

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[12] L. M. Brown and R. P. Feynman, Phys. Rev. 85, 231 (1952).

[13] When we make a change of variables, we imply of course that the spectral functions still have the same value at the same point, and not that we must take the same function of the new variables.

The modifications introduced into the theory by spin and isotopic spin are trivial. The transition amplitude will now be given by the expression

$$-A+\tfrac{1}{2}i\gamma(q_1+q_2)B, \tag{2.20}$$

and both A and B will have representations of the form (2.12). There will, further, be two amplitudes corresponding to isotopic spins of $\frac{1}{2}$ and $\frac{3}{2}$. It is sometimes more convenient to use the combinations

$$A^{(+)}=\tfrac{1}{3}(A^{(\frac{1}{2})}+2A^{(\frac{3}{2})}), \tag{2.21a}$$

$$A^{(-)}=\tfrac{1}{3}(A^{(\frac{1}{2})}-A^{(\frac{3}{2})}), \tag{2.21b}$$

and similar combinations $B^{(+)}$ and $B^{(-)}$. We then have the simple crossing relations

$$A^{(\pm)}(\nu,t)=\pm A^{(\pm)}(-\nu,t), \tag{2.22a}$$

$$B^{(\pm)}(\nu,t)=\mp B^{(\pm)}(-\nu,t), \tag{2.22b}$$

or, in terms of the spectral functions,

$$A_{13}^{(\pm)}(s,t)=\pm A_{23}^{(\pm)}(s_c,t), \tag{2.23a}$$

$$A_{12}^{(\pm)}(s,s_c)=\pm A_{12}^{(\pm)}(s_c,s), \tag{2.23b}$$

$$B_{13}^{(\pm)}(s,t)=\mp B_{23}^{(\pm)}(s_c,t), \tag{2.23c}$$

$$B_{12}^{(\pm)}(s,s_c)=\mp B_{12}^{(\pm)}(s_c,s). \tag{2.23d}$$

The poles in (2.12) and in the dispersion relations will only occur in the representation for $B^{(\pm)}$ (in pseudoscalar theory), and the second term will have a minus or plus sign in the equations for $B^{(+)}$ and $B^{(-)}$, respectively.

3. COMBINATION OF THE DISPERSION RELATIONS WITH THE UNITARITY CONDITION

The dispersion relations given in the previous section must now be combined with the unitarity equations in order to determine the scattering amplitude. We shall again begin by neglecting spin and isotopic spin; the unitarity condition (2.7) then becomes, in the one-meson approximation,

$$A_1(s,\cos\theta_1)=\frac{1}{32\pi^2}\frac{q}{W}\int \sin\theta_2 d\theta_2 d\phi_2\, A^*(s,\cos\theta_2) \times A(s,\cos(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)),$$

or

$$A_1(s,z_1)=\frac{1}{32\pi^2}\frac{q}{W}\int_{-1}^{1}dz_2\int_0^{2\pi}d\phi\, A^*(s,z_2) \times A\{s,\, z_1z_2+(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}\cos\phi\}, \tag{3.1}$$

where $z=\cos\theta$ and $\boldsymbol{\theta}_i(i=1,2)$ is a unit vector in the (θ_i,ϕ_i) direction. W is the center-of-mass energy (equal to $\sqrt{s}$), and q is the momentum in the center-of-mass system, given by the equation

$$q^2=\{s-(M+\mu)^2\}\{s-(M-\mu)^2\}/4s. \tag{3.2}$$

z is related to the momentum transfer by the simple relation

$$z=1+(t/2q^2). \tag{3.3}$$

The unitarity requirements only prove that Eq. (3.2) is true in the physical region. A_1 must then be obtained in the unphysical region by analytic continuation. In order to do this, A can be expressed as an analytic function of t or, equivalently, of z, by means of Eq. (2.11), in which the energy is kept fixed. Equation (3.3) shows that we can simply replace t by z in (2.12), so that we may write

$$A^*(s,z_2)=\frac{1}{\pi}\int dz_2'\,\frac{A_2^*(s,z_2')+A_3^*(s,z_2')}{z_2'-z_2}, \tag{3.4a}$$

$$A\{s,\, z_1z_2+(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}\cos\phi\} =\frac{1}{\pi}\int dz_3'\,\frac{A_2(s,z_3')+A_3(s,z_3')}{z_3'-z_1z_2-(1-z_1^2)^{\frac{1}{2}}(1-z_2^2)^{\frac{1}{2}}\cos\phi}. \tag{3.4b}$$

For simplicity we have included the absorptive parts A_2 and A_3 under the same integral sign, but they will of course contribute in different regions of the variable of integration. $A_2(s,z)$ will be nonzero only if $z<1-\{s-(M-\mu)^2\}/2q^2$, apart from a δ function at $z=1-(s-M^2-2\mu^2)/2q^2$, and $A_3(s,z)$ will be nonzero only if $z>1+2\mu^2/q^2$. The dispersion relations have been written down on the (incorrect) assumption that there are no subtractions necessary; we shall see in the following section how the theory must be modified to take them into account.

On substituting (3.4) into (3.2) and performing the integrations over z_2 and ϕ, we are left with the equation

$$A_1(s,z_1)=\frac{1}{16\pi^3}\frac{q}{W}\int dz_2'\int dz_3'\,\frac{1}{\sqrt{k}}\ln\frac{z_1-z_2'z_3'+\sqrt{k}}{z_1-z_2'z_3'-\sqrt{k}} \times\{A_2^*(s,z_2')+A_3^*(s,z_2')\}\{A_2(s,z_3')+A_3(s,z_3')\}, \tag{3.5}$$

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[14] H. Lehmann (to be published).

where

$$k = z_1^2 + z_2'^2 + z_3'^2 - 1 - 2z_1 z_2' z_3'. \tag{3.6}$$

We must take that branch of the logarithm which is real in the physical region $-1 < z_1 < 1$. Equation (3.5) then gives the value of A_1 in the entire complex z_1 plane.

According to Eq. (2.16), $A(s,z_1)$ must be an analytic function of t, and therefore of z, with discontinuities of magnitude $2A_{13}$ and $2A_{12}$ as z_1 crosses the positive and negative real axes. It is easily seen that the expression for A_1 in (3.5) has this property, and, on identifying the discontinuities along the real axis with A_{13} and A_{12}, we arrive at the equations

$$A_{13}(s,z_1) = \frac{1}{8\pi^2}\frac{q}{W}\int dz_2 \int dz_3\, K_1(z_1,z_2,z_3)$$
$$\times\{A_3^*(s,z_2)A_3(s,z_3) + A_2^*(s,z_2)A_2(s,z_3)\}, \tag{3.7a}$$

$$A_{12}(s,z_1) = \frac{1}{8\pi^2}\frac{q}{W}\int dz_2 \int dz_3\, K_2(z_1,z_2,z_3)$$
$$\times\{A_2^*(s,z_2)A_3(s,z_3) + A_3^*(s,z_2)A_2(s,z_3)\}. \tag{3.7b}$$

The primes on z_2 and z_3 have been suppressed. K_1 and K_2 are defined by the equations

$$\begin{aligned} &K_1(z_1,z_2,z_3) \\ &\quad = -1/[k(z_1,z_2,z_3)]^{\frac{1}{2}}, \quad z_1 > z_2 z_3 + (z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}} \\ &\quad = 0 \qquad\qquad\qquad\qquad z_1 < z_2 z_3 + (z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}} \end{aligned} \tag{3.8a}$$

$$\begin{aligned} &K_2(z_1,z_2,z_3) \\ &\quad = 1/[k(z_1,z_2,z_3)]^{\frac{1}{2}}, \quad z_1 < z_2 z_3 - (z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}} \\ &\quad = 0, \qquad\qquad\qquad\qquad z_1 > z_2 z_3 - (z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}}. \end{aligned} \tag{3.8b}$$

The points $z_1 = z_2 z_3 \pm (z_2^2-1)^{\frac{1}{2}}(z_3^2-1)^{\frac{1}{2}}$ are the points at which k changes sign.

Let us now transform back from z to our original variables. As we shall use the dispersion relations (2.17) and (2.19), it is convenient to express A_2 and A_{12} as functions of s and s_c and A_3 and A_{13} as functions of s and t. Equations (3.7) then become

$$A_{13}(s,t) = \frac{1}{32\pi^2 q^3 W}\bigg[\int dt_2 \int dt_3$$
$$\times K_1(s;\, t_1,t_2,t_3)A_3^*(s,t_2)A_3(s,t_3) \tag{3.9a}$$
$$+ \int ds_{c2}\int ds_{c3}\, K_1(s;\, t_1,s_{c2},s_{c3})A_2^*(s,s_{c2})A_2(s,s_{c3})\bigg],$$

$$A_{12}(s,s_{c1}) = \frac{1}{32\pi^2 q^3 W}\int dt_2 \int ds_{c3}\, K_2(s;\, s_{c1},t_2,s_{c3})$$
$$\times[A_3^*(s,t_2)A_2(s,s_{c3}) + A_2^*(s,s_{c3})A_3(s,t_2)]. \tag{3.9b}$$

Note that s is fixed in these equations, while s_c and t vary. K must be re-expressed as a function of the new variables by (3.3) and (2.13).

The use of Eq. (3.9), together with the dispersion relations, in order to determine the spectral functions is greatly facilitated by the fact that K is zero unless the variables satisfy certain inequalities; for all s,

$$K_1(s;\, t_1,t_2,t_3) = 0 \quad \text{unless} \quad t_1^{\frac{1}{2}} > t_2^{\frac{1}{2}} + t_3^{\frac{1}{2}}, \tag{3.10a}$$

$$K_1(s;\, t_1,s_{c2},s_{c3}) = 0 \quad \text{unless} \quad t_1^{\frac{1}{2}} > s_{c2}^{\frac{1}{2}} + s_{c3}^{\frac{1}{2}}, \tag{3.10b}$$

$$K_2(s;\, s_{c1},t_2,s_{c3}) = 0 \quad \text{unless} \quad s_{c1}^{\frac{1}{2}} > t_2^{\frac{1}{2}} + s_{c3}^{\frac{1}{2}}. \tag{3.10c}$$

(For any particular s, the restrictions on the variables could be strengthened.) Equations (3.10) are true as long as s_{c2}, s_{c3}, t_2, and t_3 are in the regions $s_c > M^2$, $t > 4\mu^2$, outside which A_2 and A_3 vanish. It follows from (3.9) that, *for any given value of t (or s_c), $A_{13}(s,t)$ [or $A_{12}(s,s_c)$] can be calculated in terms of $A_3(s,t')$ and $A_2(s,s_c')$, where the values of t' and s_c' involved are all less than t (or s_c).* On the other hand, by writing the dispersion relations (2.17) and (2.19) in the form

$$A_2(s,s_t) = \frac{1}{\pi}\int_{s_2(s_c)}^{\infty} ds' \frac{A_{12}(s',s_c)}{s'-s}$$
$$+ \frac{1}{\pi}\int_{t_1(s_c)}^{\infty} dt' \frac{A_{23}(s_c,t')}{t'-t}, \tag{3.11a}$$

$$A_3(s,t) = \frac{1}{\pi}\int_{s_3(t)}^{\infty} ds' \frac{A_{13}(s',t)}{s'-s}$$
$$+ \frac{1}{\pi}\int_{s_3(t)}^{\infty} ds' \frac{A_{23}(s',t)}{s_c'-s_c}, \tag{3.11b}$$

it is evident that $A_3(s,t)$ and $A_2(s,s_c)$ can be found in terms of $A_{12}(s',s_c)$ and $A_{13}(s',t)$, if for the moment we neglect the second term in these equations. We can therefore calculate A_{13}, A_{12}, A_3, and A_2 for all values of s and successively larger values of s_c and t. The lowest value of s_c or t for which either A_2 or A_3 is nonzero is $s_c = M^2$, at which there is a contribution of $g^2\delta(s_c - M^2)$ to A_2 from the one-nucleon state. From (3.9) and (3.10) it follows that A_{13} and A_{12} are zero if t and s_c are less than $4M^2$; for a range of values of t above this, A_{13} is nonzero and can be calculated by inserting the δ-function contribution to A_2 into (3.9a). The rest of A_2 and A_3 will still not contribute owing to (3.10). Once we have the procedure thus started, we can proceed to larger and larger values of t and s_c by alternate application of (3.9) and (3.11).[15]

Before discussing how to take the second terms of (3.11) into account, let us study in more detail the form of the functions A_{13} and A_{12} calculated thus far. In order to do this, we require the precise values of t and s_c, at a given value of s, for which the kernels K vanish; we find that

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[15] It will be noticed that, though we have brought the pole in the crossing term from the one-nucleon intermediate state into our calculations, we have not yet introduced the pole in the direct term. This pole is actually a subtraction term of Eq. (2.11) and will be treated in the following section.

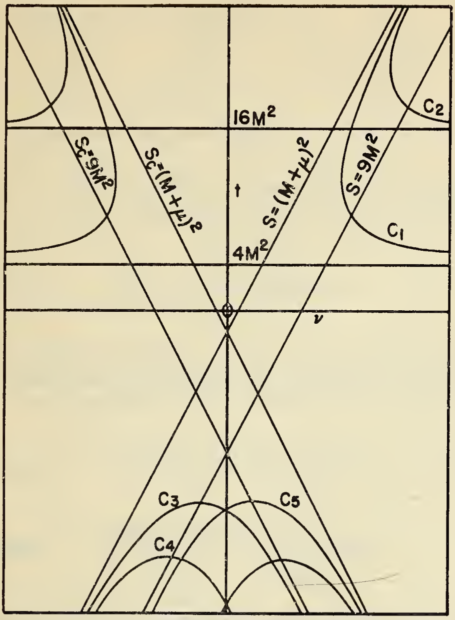

FIG. 2. Properties of the spectral functions.

$K_1(s;t_1,t_2,t_3)=0$ unless

$$t_1^{\frac{1}{2}}>t_2^{\frac{1}{2}}(1+t_3/4q^2)^{\frac{1}{2}}+t_3^{\frac{1}{2}}(1+t_2/4q^2)^{\frac{1}{2}}, \quad (3.12a)$$

$K_1(s;t_1,s_{c2},s_{c3})=0$ unless

$$t_1^{\frac{1}{2}}>(s_{c2}-u)^{\frac{1}{2}}\{1+(s_{c3}-u)/4q^2\}^{\frac{1}{2}} + (s_{c3}-u)^{\frac{1}{2}}\{1+(s_{c2}-u)/4q^2\}^{\frac{1}{2}}, \quad (3.12b)$$

$K_2(s;s_{c1},t_2,s_{c3})=0$ unless

$$(s_1-u)^{\frac{1}{2}}>t_2^{\frac{1}{2}}\{1+(s_{c3}-u)/4q^2\}^{\frac{1}{2}} + (s_{c3}-u)^{\frac{1}{2}}(1+t_2/4q^2)^{\frac{1}{2}}, \quad (3.12c)$$

where

$$u=(M^2-\mu^2)^2/s. \quad (3.13)$$

As the smallest value of s_c or t which contributes to the integrand in Eq. (3.9a) is $s_c=M^2$, where A_2 has a δ-function singularity, it follows from (3.12b) that the smallest value of t for which $A_{13}(s,t)$ is nonzero (for any given value of s) is given by

$$t^{\frac{1}{2}}=2(M^2-u)^{\frac{1}{2}}\{1+(M^2-u)/4q^2\}^{\frac{1}{2}}. \quad (3.14)$$

For very large s, this value of t approaches $4M^2$, but, as s decreases, t becomes larger and larger until, at $s=(M+\mu)^2$, it becomes infinite. Equation (3.14) has been plotted as C_1 in Fig. 2. A_{13} will be nonzero above C_1, and, near it, it will behave like $(t-t_0)^{-\frac{1}{2}}$, where t_0 is the value of t given by (3.14). It follows from (3.11b) that $A_3(s,t)$ is nonzero if $t>4M^2$, and behaves like $(t-4M^2)^{\frac{1}{2}}$ just above this limit. The value $t=4M^2$ is precisely the threshold for the process III, and we would have obtained the same results from our general reasoning in the previous section if we had neglected intermediate states containing two or more mesons but no nucleon pairs. This indicates that our assumptions are probably correct, as we have not considered the process III explicitly in this section. When we treat the subtraction terms in the dispersion relations, we shall see that A_{13} is also nonzero between $t=4\mu^2$ and $t=4M^2$, and that the region in which A_{13} is nonzero must be enlarged. The curve C_1 is therefore not yet the curve C_{13} of Fig. 1.

For a range of values of t above the curve C_1, the entire contribution to the integrand in (3.9a) comes from the δ function in A_2. At a certain point, however, the other terms in A_2 and A_3 begin to contribute. If for the moment we neglect the second term in (3.9a), the new contribution begins at the value of t obtained by putting $t_2=t_3=4M^2$ in (3.12a), since this is (at the present stage of the calculation) the lowest value of t for which A_3 is nonzero. The result has been plotted against s in Fig. 2 to give the curve C_2. As this curve approaches the line $t=16M^2$ asymptotically, there will be a corresponding new contribution to A_3 above this value, and, near it, the new contribution will behave like $(t-16M^2)^{\frac{1}{2}}$. The value $t=16M^2$ is just the threshold for the production of an additional nucleon pair in the process III, and A_3 would be expected to show such a behavior at this threshold.

We find similar discontinuities in the higher derivatives of A_{13} at series of curves (there will now be more than one for each threshold) approaching asymptotically the lines $t=4n^2M^2$, so that A_3 will have the expected behavior at the thresholds for producing n nucleon pairs.

The functions A_{12} and A_2 will exhibit the same sort of characteristics. In Eq. (3.9b), the lowest values of t_2 and s_{c3} which contribute to the integrand are $t_2=4M^2$, $s_{c3}=M^2$, so that the boundary of the region in which A_{12} is nonzero is obtained by inserting these values into (3.12c). The result is represented by the curve C_3 in Fig. 2; it approaches the line $s_c=9M^2$ as s tends to infinity. As with A_{13}, the region in which A_{12} is nonzero will be widened in the following section. From (3.19a), it follows that A_2 will (at present) be nonzero for $s_c>9M^2$, which is the threshold for pair production in the reaction II. A_{12} will also have discontinuities in the higher derivatives at series of curves such as C_4 which approach asymptotically the lines $s_c=(2n+1)^2M^2$. Finally, it can be seen that the second term of (3.9a) will give rise to further curves at which the higher derivatives of A_{13} are discontinuous, but these curves will all approach asymptotically the lines $t=4n^2M^2$.

We must now return to the second term in the Eq. (3.11), which we have so far neglected in the calculation. It can be taken into account by introducing the requirement of crossing symmetry, which has not yet been used. As in the static theory, one now has to use an iteration procedure. The function A_{23}, which only affects the crossing term in the dispersion relation (2.5), is first neglected, and the calculation done as described. A_{23} is then found from the calculated value of A_{13} and the crossing-symmetry relations (2.23), and inserted into Eq. (3.11) for the next iteration. However, the scattering amplitude calculated by this procedure would still not satisfy the equations of crossing symmetry since, while A_{13} and A_{23} are connected by (2.23a), A_{12} does not satisfy (2.23b). We have seen that the dispersion relations together with the equation of unitarity determine A_{12} uniquely, and the result is not a symmetric function of s and s_c; even the region in which it is nonzero is not symmetric. It therefore appears that we cannot satisfy simultaneously the requirements of analyticity, unitarity (in the one-meson approximation), and crossing symmetry.

The reason why this is so is easily seen in perturbation theory. Among the graphs included in the first iteration of the one-meson approximation is Fig. 3(a). The topologically similar graph Fig. 3(b) will also be included, since Fig. 3(a) by itself would have square roots in the energy denominators and would not have the necessary analytic properties. If, therefore, crossing symmetry is to be maintained, Fig. 3(c) must also be included. In this graph, however, there is an intermediate state of a nucleon and a pair, so that the unitarity condition in the one-meson approximation is not satisfied.

This example also indicates how we should modify our iteration procedure. In addition to inserting a term A_{23}, obtained by crossing symmetry from the previous iteration, into (3.11), we must insert a term $A_{12}'(s,s_c)$ equal to $A_{12}(s_c,s)$ as calculated in the previous iteration. The contribution from this term is to be added to the contribution from $A_{12}(s,s_c)$ calculated in the normal way. A_{12}' will be nonzero above the curve C_5 in Fig. 2, and, in particular, it will be zero for all values of s_c if s is less than $9M^2$. Complete crossing symmetry is now maintained, but the addition of A_{12} violates the unitarity condition (in the one-meson approximation) for values of s greater than $9M^2$, and a perturbation expansion would include graphs such as Fig. 3(c). As these graphs will appear in higher approximations, the fact that we are forced to include them here should not be considered a disadvantage of our method. In any case, the unitarity condition is only violated where the one-meson approximation is far from correct.

The iteration procedure is found to give rise to further curves, like C_2 and C_4 (Fig. 2), at which the higher derivatives of the spectral functions are discontinuous. These new discontinuities correspond to the production of mesons together with nucleon pairs. We still do not have discontinuities at all possible thresholds.

The inclusion of the spin does not change any of the essential features of the theory, though the details are

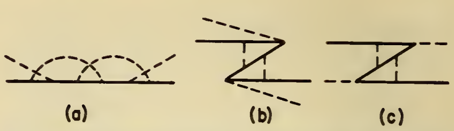

FIG. 3. Graphs which bring in intermediate states with pairs.

rather more complicated. Following Chew *et al.*,[1] we write the pion-nucleon T matrix in the form

$$T=-\frac{2\pi W}{Ew}(a+\sigma\cdot\mathbf{q}_2\sigma\cdot\mathbf{q}_1 b), \tag{3.15}$$

where E is the center-of-mass energy of the nucleon and w that of the pion. a and b are related to the quantities A and B in the expression (2.20) by the formulas

$$a=\frac{E+M}{2W}\left(\frac{A+(W-M)B}{4\pi}\right), \tag{3.16a}$$

$$b=\frac{E-M}{2W}\left(\frac{-A+(W+M)B}{4\pi}\right). \tag{3.16b}$$

The unitarity condition corresponding to (3.7) can now be worked out in terms of a and b; the equation obtained is

$$a_{13(12)}(s,z_1)=\sum_{\alpha}\frac{q}{\pi}\int dz_2\int dz_3\, K_{1(2)}(z_1,z_2,z_3)$$
$$\times\left\{a_\alpha^*(s,z_2)a_\alpha(s,z_3)+\frac{z_2-z_3z_1}{1-z_1^2}b_\alpha^*(s,z_2)a_\alpha(s,z_3)\right.$$
$$\left.+\frac{z_3-z_2z_1}{1-z_1^2}a_\alpha^*(s,z_2)b_\alpha(s,z_3)\right\}, \tag{3.17a}$$

$$b_{13(12)}(s,z_1)=\sum_{\alpha}\frac{q}{\pi}\int dz_2\int dz_3\, K_{1(2)}(z_1,z_2,z_3)$$
$$\times\left\{\frac{z_3-z_2z_1}{1-z_1^2}b_\alpha^*(s,z_2)a_\alpha(s,z_3)+\frac{z_2-z_3z_1}{1-z_1^2}\right.$$
$$\left.\times a_\alpha^*(s,z_2)b_\alpha(s,z_3)+b_\alpha^*(s,z_2)b_\alpha(s,z_3)\right\}, \tag{3.17b}$$

where $\sum_\alpha$ indicates that terms of the form $a_\alpha^*a_\alpha$ are to be replaced by $a_2^*a_2+a_3^*a_3$ in the calculation of a_{13} and b_{13} and by $a_2^*a_3+a_3^*a_2$ in the calculation of a_{12} and b_{12}, exactly as in (3.7). a_2 and b_2, a_3 and b_3, a_{12} and b_{12}, and a_{13} and b_{13} are related respectively to A_2 and B_2, A_3 and B_3, A_{12} and B_{12}, and A_{13} and B_{13} by Eqs. (3.16). The unitarity condition (3.17) can be rewritten in terms of A and B; it then becomes

$$A_{13(12)}(s,z_1)=\sum_\alpha \frac{q}{4\pi^2 W}\int dz_2\int dz_3\, K_{1(2)}(z_1,z_2,z_3)$$
$$\times\left\{\left(1-\frac{w}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\right)A_\alpha{}^*(s,z_2)A_\alpha(s,z_3)\right.$$
$$+\left(\frac{\omega}{2}\frac{1-z_2+z_3-z_1}{1-z_1}+\frac{Mw}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\right)$$
$$\times A_\alpha{}^*(s,z_2)B_\alpha(s,z_3)+\left(\frac{\omega}{2}\frac{1+z_2-z_3-z_1}{1-z_1}\right.$$
$$\left.+\frac{Mw}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\right)B_\alpha{}^*(s,z_2)A_\alpha(s,z_3)$$
$$\left.+\frac{W^2-M^2}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}B_\alpha{}^*(s,z_2)B_\alpha(s,z_3)\right\},\quad (3.18a)$$

$$B_{13(12)}(s,z_1)=\sum_\alpha \frac{q}{4\pi^2 W}\int dz_2\int dz_3\, K_{1(2)}(z_1,z_2,z_3)$$
$$\times\left\{\frac{E}{2MW}\frac{1-z_2-z_3+z_1}{1+z_1}A_\alpha{}^*(s,z_2)A_\alpha(s,z_3)\right.$$
$$+\left(\frac{1+z_2-z_3-z_1}{2(1-z_1)}-\frac{E}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\right)$$
$$\times A_\alpha{}^*(s,z_2)B_\alpha(s,z_3)+\left(\frac{1-z_2+z_3-z_1}{1-z_1}\right.$$
$$\left.-\frac{E}{2W}\frac{1-z_2-z_3+z_1}{1+z_1}\right)B_\alpha{}^*(s,z_2)A_\alpha(s,z_3)$$
$$+\left(\omega-\frac{(w^2-M^2)E}{2MW}\frac{1-z_2-z_3+z_1}{1+z_1}\right)$$
$$\left.\times B_\alpha{}^*(s,z_2)B_\alpha(s,z_3)\right\}.\quad (3.18b)$$

Equations (3.17) and (3.18) will hold separately for the amplitudes corresponding to isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$.

It remains to justify the claim that the result calculated by our procedure, if expanded in a perturbation series, would give a subset of the usual perturbation series. The proof is somewhat awkward because we were unable to satisfy the unitarity condition in the one-meson approximation at all values of the energy. Let us first ignore this. The nth term in the perturbation series $A^{(n)}$ is then determined uniquely in the physical region by the following two requirements:

(i) For sufficiently small values of the momentum transfer $\{$less than $2\mu[\frac{2}{3}(2M+\mu)/(2M-\mu)]^{\frac{1}{2}}\}$, $A^{(n)}$ must satisfy the dispersion relation (2.5), a result which has been proved rigorously.[14] The absorptive part A_1 (and hence, by crossing symmetry, A_2) is known, since it is determined by unitarity in terms of lower order perturbation terms in the physical region, and by analytic continuation (with s constant) outside it.[14]

(ii) For a fixed value of s, $A^{(n)}$ is an analytic function of the momentum transfer throughout the physical region.[14]

As the functions calculated by our method certainly fulfil these requirements, they must generate the correct perturbation series.

However, our result does not satisfy the unitarity condition in the one-meson approximation at all energies, and we must examine more closely how A_1 is to be determined. Let us assume that our method gives the correct perturbation series up to the $(n-1)$th order. The reasoning developed in this section then shows that the nth-order contribution to A_1 will be of the form

$$A_1{}^{(n)}=\frac{1}{\pi}\int dt'\,\frac{A_{13}{}^{(n)}(s,t')}{t'-t}-\frac{1}{\pi}\int dt'\,\frac{A_{12}{}^{(n)}(s,t')}{t'-t},\quad (3.19)$$

where $A_{13}{}^{(n)}$ and $A_{12}{}^{(n)}$ are certainly zero below C_1 and above C_3, respectively, in Fig. 2. Inserting this expression into (2.5), we find that

$$A_d{}^{(n)}=\frac{1}{\pi^2}\int ds'\int dt'\,\frac{A_{13}{}^{(n)}(s',t')}{(s'-s)(t'-t)}$$
$$-\frac{1}{\pi^2}\int ds'\int dt'\,\frac{A_{12}{}^{(n)}(s',t')}{(s'-s)(t'-t)}.\quad (3.20)$$

The suffix d indicates that we are considering the direct and not the crossing term. The second term of (3.20) will not be an analytic function of t in the physical region, but it will have a branch point at the largest value of t for which A_{12} is nonzero. We can make it analytic by adding to A_2 the expression

$$-\frac{1}{\pi}\int dt'\,\frac{A_{12}{}^{(n)}(s_c,t')}{t'-t},\quad (3.21)$$

which we would expect from (2.17), if our representation is correct. By inserting this into (2.5) and adding the result to the second term of (3.20), we obtain

$$\frac{1}{\pi^2}\int ds'\int ds_c{}'\,\frac{A_{12}{}^{(n)}(s',s_c{}')}{(s'-s)(s_c{}'-s_c)},\quad (3.22)$$

which is analytic in the physical region. The contribution (3.21) to $A_2{}^{(n)}$ is uniquely determined from the requirement that $A^{(n)}$ be an analytic function of the momentum transfer in the physical region, and is nonzero only for $s_c>9M^2$. It corresponds to adding a graph such as Fig. 3(b) to Fig. 3(a); as A_1 for Fig. 3(c) is nonzero for $s>9M^2$, A_2 for Fig. 3(b) will be nonzero for $s_c>9M^2$.

Finally, then, the *n*th-order perturbation term can be determined from the lower order perturbation terms without using any unproved properties of the scattering amplitude as follows:

(i) Calculate A_1 by unitarity, and extend it into the nonphysical region for momentum transfers less than $2\mu[\frac{2}{3}(2M+\mu)/(2M-\mu)]^{\frac{1}{2}}$ by analytic continuation.

(ii) Calculate a contribution $A_{2d}^{(n)}$ to $A_2^{(n)}$, for $s_c>9M^2$, from the requirement that if it, together with A_1, be inserted into (2.5), the resulting function $A_d^{(n)}$ must be an analytic function of the momentum transfer in the physical region. By doing this we partially include intermediate states with nucleon pairs, which is necessary if we are to maintain the required analytic properties and crossing symmetry.

(iii) Now calculate $A_2^{(n)}$ and the extra contribution to $A_1^{(n)}$ by crossing symmetry from $A_1^{(n)}$ and the extra contribution to $A_2^{(n)}$.

(iv) Find $A^{(n)}$ from (2.5) for values of the momentum transfer less than $2\mu[\frac{2}{3}(2M+\mu)/(2M-\mu)]^{\frac{1}{2}}$, and calculate it in the rest of the physical region by analytic continuation in t.

This procedure defines a one-meson approximation in perturbation theory. From what has been said, it is clear that our solution will give precisely this perturbation expansion, so that our assumptions are justified in perturbation theory.

4. SUBTRACTION TERMS IN THE DISPERSION RELATIONS

We have thus far assumed that the dispersion relations are true without any subtractions. As we have pointed out in the first section, by doing this we neglect what is physically the most important part of the scattering amplitude. In this section we shall investigate how many subtractions are necessary for each dispersion relation and shall outline how they can be calculated, leaving the details for a further paper.

Let us first consider Eqs. (2.11) and (2.16), which were used in obtaining the unitarity condition (3.9) [or (3.18) for nucleons with spin]. Even if these dispersion relations are written with subtraction terms, it is found that (3.9) is unchanged, so that the subtraction terms are only needed in the final evaluation of A from A_2 and A_3 by means of (2.11), or of A_1 from A_{12} and A_{13} by means of (2.16). The number of subtractions will depend on the behavior of A_{12}, A_{13}, A_2, and A_3, as calculated by our procedure, as s_c and t tend to infinity—we shall have to perform at least enough subtractions for (2.11) and (2.16) to converge.

It is difficult to make an estimate of the behavior of these functions at infinite values of s_c and t from the equations determining them, and we shall use indirect arguments which, though not rigorous, are very plausible. We shall find that, if the coupling constant is small enough, the functions tend to zero at infinity, so that one can write the dispersion relations without any subtractions. For larger values of the coupling constant, more and more subtractions will be needed. The reader who is prepared to accept this may omit the following two paragraphs.

We consider only the first iteration, since subsequent iterations proceed in a similar way and the results are unlikely to be qualitatively different. The result can then be expanded in a perturbation series. If the solutions obtained for this problem by other methods, such as the Tamm-Dancoff or Bethe-Salpeter methods, are expanded in a perturbation series, it is found that the series for each angular momentum state converges as long as the coupling constant is within a certain radius of convergence, and that this radius of a convergence tends to infinity with the angular momentum.[16] Our perturbation series would be different from the perturbation series obtained by these methods, partly because the intermediate states with pairs which we include are not the same as those included by either of them, and partly because, in calculating the subtraction terms (other than those at present under discussion), we shall not take into account terms corresponding to all graphs included by these approximations. Such differences would not be expected to affect qualitatively the convergence properties of the angular momentum states, and we shall assume that the results quoted above are true for our perturbation series too.

The transition amplitude for the state of total angular momentum j and orbital angular momentum $j\pm\frac{1}{2}$ can be shown to be

$$f_{j\pm}=\int_{-1}^{1}dz\,a(s,z)P_{j\pm\frac{1}{2}}(z)+\int_{1}^{1}dz\,b(s,z)P_{j\mp\frac{1}{2}}(z),\quad(4.1)$$

where a and b are the functions defined in (3.15) and (3.16). Now it is easily seen that each term in the perturbation series for $a_2(s,z)$, $a_3(s,z)$, $b_2(s,z)$, and $b_3(s,z)$ tends to zero like $1/z$ as z tends to infinity, so that the dispersion relation (2.11) for each term can be written down without any subtractions. Hence

$$\begin{aligned}f_{j\pm}^{(n)}&=\int_{-1}^{1}dz\int dz'\left\{\frac{a_2^{(n)}(s,z')+a_3^{(n)}(s,z')}{z'-z}P_{j\pm\frac{1}{2}}(z)\right.\\&\qquad\left.+\frac{b_2^{(n)}(s,z')+b_3^{(n)}(s,z')}{z'-z}P_{j\mp\frac{1}{2}}(z)\right\}\quad(4.2)\\&=\int dz'\{[a_2^{(n)}(s,z')+a_3^{(n)}(s,z')]\phi_{j\pm\frac{1}{2}}(z')\\&\qquad+[b_2^{(n)}(s,z')+b_3^{(n)}(s,z')]\phi_{j\mp\frac{1}{2}}(z')\},\quad(4.3)\end{aligned}$$

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[16] Note that the "potential" in the Tamm-Dancoff or Bethe-Salpeter equation involved includes only the crossing term and not the direct term, which has still to be brought into the calculation.

where

$$\phi_n(z')=\int_{-1}^{1}dz\,\frac{P_n(z)}{z'-z}$$
$$\approx 1/z'^{n+1}\quad\text{as}\quad z'\rightarrow\infty. \tag{4.4}$$

Let us suppose that the value of the coupling constant is such that the perturbation series for states of angular momentum j_1 converges. If each term in the perturbation series for this angular momentum state is expressed by (4.3), and if we assume that we can interchange the order of summation and integration, we arrive at the equation

$$f_{j1\pm}=\int dz'\{\textstyle\sum_n[a_2^{(n)}(s,z')+a_3^{(n)}(s,z')]\phi_{j1\pm\frac{1}{2}}(z')$$
$$+\textstyle\sum_n[b_2^{(n)}(s,z')+b_3^{(n)}(s,z')]\phi_{j1\mp\frac{1}{2}}(z')\}. \tag{4.5}$$

In order for the integrand to exist, we see from (4.4) that a and b must be smaller than $z^{j_1-\frac{1}{2}}$ at infinite z. The dispersion relations can therefore be written down with not more than $j-\frac{1}{2}$ subtractions. In particular, if the coupling constant is small enough the dispersion relations can be written down without any subtractions.[17]

If the coupling constant is such that n subtractions are required, the unitarity condition for the states of angular momentum $\frac{1}{2}$ to $n-\frac{1}{2}$ will have to be applied separately. The wave functions for these states are polynomials of degree not greater than $n-1$ in the variable z (or s_c and t), and are not determined from the absorptive parts in the dispersion relations (2.11) and (2.16).

The calculation must be done after each iteration, as the result will be needed for the next iteration. The details of the calculation will not be discussed here, but they will in principle be similar to those of Chew and Low[4] and Dalitz, Castillejo, and Dyson,[5] and will involve considering the reciprocal of the scattering amplitude. The analytic properties of the individual angular momentum states are not as simple as in the static theory, but they can be determined from the assumed analytic properties of the transition amplitude, and, as in the static theory, the singularities not on the positive real axis can be found from the previous iteration.

The precise number of subtractions required cannot be determined without calculating the result, but it is almost certainly not less than two. It is difficult to see how the observed resonant behavior of the $P_{\frac{3}{2}}$ state could be reproduced by means of the calculations described in the last section, whereas it follows quite naturally from a Chew-Low-type calculation. If the coupling constant were large enough to bind the (3,3) resonance state, and for a certain range of values of the coupling constant below this, we would definitely have to perform two subtractions. The precise range involved is difficult to determine, but it would be expected to include those values of the coupling constant for which the (3,3) state still has the appearance of an unstable isobar. Until we state otherwise, however, we shall suppose that the coupling constant is sufficiently small for the functions $A(s,z)$ and $B(s,z)$ to tend to zero at infinite z, as the situation with regard to the other subtractions is much simpler in this case. Even then, we would have to perform one subtraction for each of A and B, since the calculations of the previous section did not include the pole of the scattering amplitude from the one-nucleon intermediate state; only the pole in the crossing term was included. The pole affects the states with $j=\frac{1}{2}$ alone, so that, if we apply the unitarity condition for these states separately by the Chew-Low method, we can include it correctly. We thereby change A and B by a quantity independent of z.

When we calculate the scattering amplitudes for the states with $j=\frac{1}{2}$, we find a ghost state in the first iteration, just as in all other models. In subsequent iterations, however, where the crossing terms contribute, it does not follow from the form of the equations that we shall necessarily find a ghost state, and, judging from the charged scalar model, we may hope that the ghost state does not in fact occur.

We now turn to consider the subtraction terms in the other dispersion relations used in the calculations, Eq. (3.11). By putting the δ-function contribution to A_2 into (3.18), it can be seen that the lowest order term in $A_{13}(s,t)$ tends to a constant as s tends to infinity, whereas the lowest order term in $B_{13}(s,t)$ behaves like $1/s$. For a certain range of values of t, only the lowest order term contributes to A_{13} and B_{13}, so that there will certainly be one subtraction in Eq. (3.11b) for A_3, while the equation for B_3 could be written down without any subtractions. We find similarly that both $A_{12}(s,s_c)$ and $B_{12}(s,s_c)$ tend to zero like $1/s$ as s tends to infinity. It would therefore appear that the dispersion relations (3.11a) did not require any subtractions. However, we have seen that $A_1(s,s_c)$ and $B_1(s,s_c)$ behave like a constant for large s_c with s constant, even for small values of the coupling constant, so that, by crossing symmetry, $A_2(s,s_c)$ and $B_2(s,s_c)$ will behave like a constant for large s. There will therefore be one subtraction term in Eqs. (3.11a) for both A_2 and B_2.

The determination of the subtraction terms in Eq. (3.11a) is not difficult, since the contributions to A_2 and B_2 from the states with $j=\frac{1}{2}$ (with the energy s_c of the reaction II kept constant) can be found by crossing symmetry from the corresponding contributions to A_1 and B_1 in the previous iteration. However, for the subtraction terms in Eq. (3.11b), we require the unitarity condition for A_3, which involves the reaction III. As there is one subtraction, only the S waves will be involved. Again we have to limit the intermediate states considered; in this first approximation we would consider the two-meson states ("two-meson approximation") and perhaps the nucleon-antinucleon intermediate states ("two-meson plus pair approximation") as well. We shall then require the meson-meson scattering amplitude (and the nucleon-antinucleon scattering amplitude if nucleon-antinucleon intermediate states are being considered). The determination of these scattering amplitudes would be as extensive a calculation as the determination of the pion-nucleon scattering amplitude, but neglect of the crossing term would probably not give rise to too great an error in our final result, in which case the S-wave amplitudes could be written down immediately in the two-meson or two-meson plus pair approximations. The meson-meson coupling constant is thereby introduced into the calculation, as has been mentioned in the introduction. Once the meson-meson and nucleon-antinucleon scattering amplitudes are known, the transition amplitude for the reaction III can be calculated. Since the integral equation is now linear, the details will be different from those of the Chew-Low calculations, but, as in their case, the solution could be written down exactly if there were no other singularities of the transition amplitude, and we can use an iteration procedure for the actual problem. The iterations will again be interspersed between the iterations of the main calculation. The S-wave portion of A_3, as calculated by this procedure, will be nonzero for $t>4\mu^2$, so that the scattering amplitude now has the expected spectral properties. The boundaries of the regions in which the spectral functions are nonzero will thereby also be changed; this will be discussed in more detail at the end of the section.

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[17] We should emphasize that it is only in the first iteration that we relate the number of subtractions needed to the convergence of the angular momentum states. We say nothing at all about the convergence of the perturbation series in subsequent iterations, but assume simply that the behavior of the spectral functions at infinite values of z is not likely to be qualitatively different from their behavior in the first iteration.

We have seen that, as long as the coupling constant is sufficiently small, we require one subtraction for each of the dispersion relations except the dispersion relation (3.11b) for B_3, for which we do not require any subtractions. It is also easily seen that this behavior is consistent—the functions as calculated in the last section, with the calculations modified by the subtraction terms, will not at any stage become too large at infinity. If, however, one were to make any additional subtractions, one would find that, on performing the calculations, one would need more and more subtractions as the work progressed, and one could not obtain any final result. The number of subtractions to be performed is therefore determined uniquely. There is one exception to this statement: we could perform one subtraction in Eq. (3.11b) for B_3. Such a subtraction is, however, excluded by the requirement that the theory remain consistent when the interaction with the electromagnetic field is introduced. If one were to make this subtraction, the scattering amplitude would behave like $f(t)\gamma(q_1+q_2)$ for large values of s. It then follows from gauge invariance that the matrix element for the processes

$$\pi^{\pm}+n\rightarrow\pi^{\pm}+n+\nu \quad\text{or}\quad \pi^0+p\rightarrow\pi^0+p+\nu$$

will contain a term which behaves like $f(t)\gamma$ for large s, where t is now minus the square of the momentum transfer of the neutral particle.[18] The contribution to B_1 and B_{13} from the $\pi-N-\gamma$ intermediate state therefore tends to infinity at least as fast as s for infinite s, so that one would require two subtractions for the dispersion relation in question and the theory would not be consistent.

Since the unitarity conditions for the two $j=\frac{1}{2}$ states of the pion-nucleon system, and for the S state of the pion-pion system, have to be applied separately by the Chew-Low method, there will be Castillejo-Dalitz-Dyson ambiguities associated with these states. The ambiguities will of course affect all states in subsequent iterations. They correspond to the existence of unstable baryons of spin $\frac{1}{2}$ and either parity, or of heavy unstable mesons of spin zero. There are no ambiguities associated with states of higher angular momentum; this is in agreement with perturbation theory, according to which it is impossible to renormalize systems containing particles of spin 1 or more. Had there been no interaction with the electromagnetic field, we could have introduced a further subtraction term which would have necessitated a separate application of the unitarity condition for the P state of the pion-pion system. The resulting Castillejo-Dalitz-Dyson ambiguity would have been associated with a heavy unstable meson of spin 1. This corresponds to the Bethe-Beard mixture of vector and scalar mesons, which can be renormalized in perturbation theory as long as there is no interaction with the electromagnetic field.

Now let us consider the situation that occurs in practice, when the coupling constant is sufficiently large for the scattering amplitude and its absorptive parts to tend to infinity with z (or s_c and t) when s remains constant. The function A_{12}' which, according to our procedure, must be added to A_{12} in iterations other than the first, will now tend to infinity with s, so that A_2, as calculated from (3.11a), would show a similar behavior. In practice, when the unitarity condition for states with $j=\frac{3}{2}$ as well as with $j=\frac{1}{2}$ must be applied separately, $A_{12}'(s,s_c)$ and $A_{23}(s_c,t)$ will tend to infinity faster than s or t, and the dispersion relation (3.11a) will require two subtractions. The subtraction terms can be determined by crossing symmetry as before. However, we have seen that, if A_2 tends to infinity with s, we cannot consistently perform the calculation, so that we shall have to introduce some further modifications.

The reason for the difficulty is probably the inadequacy of the one-meson approximation. The breakdown occurs just at the value of the coupling constant for which the contribution to the scattering amplitude from A_{12}' is comparable to the remainder of the scattering amplitude when s is large. Since that part of A_1 calculated from A_{12}' represents a partial effect of states with one or more pairs, the contribution of these intermediate states is now important at high energies and it seems reasonable that, if one could take them into account properly, one could still perform the calculations for large values of the coupling constant. In the one-meson approximation, one would have to make some sort of a cutoff to the contribution to A_2 from the crossing term above $s=9M^2$. As this entails modifying the unitarity condition in the region where it is in any case inaccurate, it is consistent with our approximations, and it may be hoped that the theory is not very sensitive to the precise location and form of the cutoff. If one were to go to further approximations in which intermediate states with pairs were included, the cutoff would always be applied only at or above the threshold for processes which were neglected.

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[18] This can be shown by using a generalization of the Ward identity due to H. S. Green, Proc. Phys. Soc. (London) 66, 873 (1953), and T. D. Lee, Phys. Rev. 95, 1329 (1954), and proved by Y. Takahashi, Nuovo cimento 6, 372 (1957).

Once we are prepared to introduce cutoffs into our approximations, we might legitimately ask whether or not we should perform more than one subtraction in Eq. (3.11b). This could only be determined by examining the behavior of the scattering amplitude and its absorptive parts at large values of s when we go beyond the one-meson approximation. However, if A and B have the behavior assumed thus far (A remains constant and B behaves like $1/s$), the cross section would tend to zero like $1/s$ at large s, whereas the experimental results indicate that the cross section remains constant. It therefore may be necessary to perform an additional subtraction and to introduce the unitarity condition of the reaction III in P states.

At first sight it would seem as though there were Castillejo-Dalitz-Dyson ambiguities associated with all states for which the unitarity condition has to be applied separately, not only with the $j=\frac{1}{2}$ states. However, it is also possible that only the solution without any of the extra terms in the higher angular momentum waves would converge as we introduced more and more states into the unitarity equations. This solution would be an analytic continuation of the solution obtained for small values of the coupling constant, whereas the other solutions could not be continued below a certain value of the coupling constant and would have no perturbation expansion. While we can by no means exclude such a behavior, it nevertheless gives us grounds to suppose that the ambiguity exists only for meson-nucleon states with $j=\frac{1}{2}$ and for S-wave meson-meson states, even when the coupling constant is large.

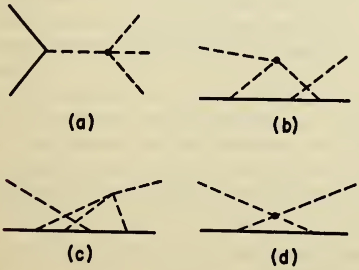

FIG. 4. Graphs involving the pion-pion interaction.

Before leaving this section, let us state the boundaries of the region in which the spectral functions A_{13}, A_{23}, and A_{12} are nonzero, i.e., the position of the curves C_{13}, C_{23}, and C_{12} in Fig. 1. Since A_3 is now nonzero for $t^2>4\mu^2$, C_{13} in the one-meson approximation is obtained by putting $t_2=t_3=4\mu^2$ in (3.12a), so that

$$t_{1a}^{\frac{1}{2}}=4\mu(1+\mu^2/q^2)^{\frac{1}{2}},$$

or

$$t_{1a}=\frac{16\mu^2(s-M^2+\mu^2)^2}{[s-(M+\mu)^2][s-(M-\mu)^2]}. \tag{4.6}$$

For any given value of s, A_{13} will be nonzero if $t>t_{1a}$. We notice that, as s tends to infinity, t_{1a} approaches the value $16\mu^2$. This is not the expected result—we have shown in Sec. 2 that it should approach the value $4\mu^2$. The reason for the discrepancy is that, in our approximation, the reaction III takes place purely through S waves for $4\mu^2<t<16\mu^2$, and A_3 will be a function only of t in this region. Had it been possible for the reaction III to go through an intermediate state of one pion, A_3 would have had a δ function at $t=\mu^2$, and, on putting this value into (3.12a), we would have obtained the expected result. As it is, however, we shall have to go beyond the one-meson approximation to get the correct boundary of A_{13}.

The reaction $N+\bar{N}\rightarrow 3\pi$ can go through a one-pion intermediate state by means of the process represented in Fig. 4(a). If, therefore, we treat the outgoing pions in the reaction $N+\pi\rightarrow N+2\pi$ as one particle with fixed energy and angular momentum, and represent the transition amplitude in the same way as we have represented the transition amplitude for pion-nucleon scattering, the absorptive part corresponding to A_3 will have a δ function at $t=\mu^2$. We can work out the resulting contribution to A_{13} (of the pion-nucleon scattering amplitude) by unitarity in the same way as we worked out the contributions from the one-meson approximation. z_2 and z_3 in Eqs. (3.4)–(3.8) will now refer to the center-of-mass deflection of the nucleon in the production reaction, and will be connected with the momentum transfer by the relation

$$z=\{q^2+q_1^2+t-[(M^2+q^2)^{\frac{1}{2}}-(M^2+q_1^2)^{\frac{1}{2}}]^2\}/2qq_1,$$

where q_1, is the center-of-mass momentum of the outgoing nucleon. The value of q_1 will depend on the relative energy of the two pions; we shall require the maximum value of q_1 (for a fixed s), which occurs when the pions are at rest with respect to one another and is given by

$$q_{1m}{}^2=\{s-(M+2\mu)^2\}\{s-(M-2\mu)^2\}/4s. \quad (4.7)$$

We then find that the boundary of this contribution to A_{13} has the equation

$$t_{1b}=\frac{4\mu^2(s-M^2-2\mu)^2}{[s-(M+2\mu)^2][s-(M-2\mu)^2]}. \quad (4.8)$$

The curve represented by (4.8) approaches asymptotically the lines $t=4\mu^2$ and $s=(M+2\mu)^2$. Thus, as would be expected, this contribution to A_{13} only occurs above the threshold for pion production.

A_{13} is therefore nonzero for $t>t_1$, where

$$\begin{aligned} t_1&=t_{1a}, & (M+\mu)^2<s<(M+2\mu)^2;\\ t_1&=\min(t_{1a},t_{1b}), & (M+2\mu^2)<s<\infty; \end{aligned} \quad (4.9)$$

and $t=t_1$ is the curve C_{13} of Fig. 1. We cannot be sure that contributions from other intermediate states will not extend beyond this curve, but this is unlikely owing to the greater mass of these states.

The curve C_{23} is obtained from C_{13} simply by changing s to s_c. C_{12} can be calculated in a similar way; we find that

$$\begin{aligned} s_{c2}&=s_{c2a}, & (M+\mu)^2<s<(M+2\mu)^2\\ &=\min(s_{c2a},s_{c2b}), & (M+2\mu)^2<s<\infty, \end{aligned} \quad (4.10)$$

where

$$(s_{c2a}-u)^{\frac{1}{2}}=2\mu\left\{\frac{s^2-s(3M^2+2\mu^2)+2(M^2-\mu^2)^2}{[s-(M+\mu)^2][s-(M-u)^2]}\right\}^{\frac{1}{2}}$$
$$+\left\{\frac{[M^2s-(M^2-\mu^2)^2][s^2-2s(M^2+3\mu^2)+(M^2-\mu^2)^2]}{s[s-(M+\mu)^2][s-(M-\mu)^2]}\right\}^{\frac{1}{2}}, \quad (4.11)$$

$$s_{c2b}(s)=s(s_{c2a}). \quad (4.12)$$

The equation $s_a=s_{c2b}$ represents in fact the boundary of the region in which A_{12}' is nonzero. We observe that, once the pion-pion interaction has been included, this region approaches asymptotically the line $s=(M+2\mu)^2$ rather than the line $s=9M^2$. The reason is that processes represented by graphs such as Fig. 4(b) are now included in our approximation, so that the crossing term will include the contribution from Fig. 4(c), the intermediate state of which involves a nucleon and two pions.

For a given real value of s, the absorptive part A of the scattering amplitude will be an analytic function of the momentum transfer as long as

$$t_2<t<t_1, \quad (4.13a)$$

where t_1 is given by (4.9), and t_2 by (4.10) and (2.13). The expansion in partial waves will converge if

$$-t_1-4q^2<t<t_1, \quad (4.13b)$$

as $-t_1-4q^2$ is always greater than t_2.

We may note finally one interesting point concerning the spectral properties of the scattering amplitude. The unitarity condition should, strictly, be used in the physical region only, and the results extended to the unphysical region by analytic continuation. This has actually been done for the reaction I, as well as for the reaction III with $t>4M^2$. For the reaction III in the region $4\mu^2<t<4M^2$, we should apply the unitarity condition with the nucleon masses taken, not on the mass shell, but at some smaller value where all the momenta would be real. The result should then be continued analytically onto the mass shell. In our case this is found to make no difference, but if, in addition to the nucleon, we had a baryon whose mass M_B satisfied the inequality

$$M_B{}^2<M^2-\mu^2, \quad (4.14)$$

it would be necessary to do the calculation in this way. On making the continuation to the mass shell, it would be found that the absorptive part A_3 extended below the limit $t^2=4\mu^2$. It has been shown by several workers[19] that, if an inequality such as (4.14) is satisfied, the vertex function would show similar spectral properties. The simplest graph to exhibit them in our case would be Fig. 4(d), which will obviously have properties similar to those of a vertex graph. It is thus seen that these spectral abnormalities would not limit the applicability of our method, but, on the contrary, follow from it.

5. APPROXIMATION SCHEME FOR OBTAINING THE SCATTERING AMPLITUDE

In the methods developed in the previous sections, the unitarity condition for the reaction I is satisfied for all angular-momentum states in the one-meson approximation. The unitarity condition for the reaction III is satisfied only for S states in the two-meson or two-meson plus pair approximations. The unitarity condition for higher angular momentum states of the reaction III is not satisfied, but the scattering amplitude shows the expected behavior at the threshold for competing real processes.

These properties suggest immediately a further approximation which would be consistent with our other approximations. The major portion of the work, and certainly the major part of the computing time, would be employed in calculating the spectral functions, as this involves finding double integrals which are themselves functions of two variables. The calculations would therefore be simplified if we neglected those contributions to the spectral functions which begin at the threshold for processes involving more than two particles. The only contributions to A_{13} and A_{23} left would be those beginning at $t=4M^2$, and they could be obtained by inserting the δ-function contribution to B_2

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[19] Karplus, Sommerfield, and Wichman, Phys. Rev. 111, 1187 (1958); Y. Nambu, Nuovo cimento 9, 610 (1958); R. Oehme, Phys. Rev. 111, 1430 (1958).

into (3.18). The spectral function A_{12} would be zero in this approximation.

The unitarity condition for the higher angular momentum states of the reaction I is no longer satisfied. However, the terms neglected appear by their form to arise from intermediate states of the reaction III with more than two particles, so that the approximation is in the spirit of the approximations already made. We have in fact made precisely this approximation in the unitarity condition for the S waves of the reaction III. The unitarity condition for the low angular momentum states of the reaction I, and in particular for the states with $j=\frac{1}{2}$ or $\frac{3}{2}$, is still satisfied, as it has been introduced separately. The present approximation treats the reactions I, II, and III on the same footing.

To summarize, then, our method of procedure will be the following: The first few angular momentum states of A_1 and A_3 are found on the assumption that each angular momentum state is an analytic function of the square of the center-of-mass energy except for the perturbation singularities and the cuts on the positive real axis. This calculation can be done exactly if the discontinuity across the cut along the positive real axis is determined by unitarity (complications arise, as the relations connecting a and b with A and B involve square roots of kinematical factors, but the methods can be modified accordingly). A_{13} and A_{12} are also found as just described. The analytic properties of the low angular momentum states are now determined from the analytic properties of the scattering amplitude given by (2.12). The singularities can be calculated in terms of A_1, A_2, A_3. These absorptive parts can in turn be found from A_{13} and A_{23} by means of the dispersion relations (2.16), (2.17), (2.19), with subtraction terms which can be obtained from the low-angular-momentum states. In the next iteration, all the singularities of the low angular momentum states except that along the positive real axis are found from the quantities calculated in the first iteration, and the singularity along the positive real axis is redetermined from the unitarity condition. The iteration procedure is repeated until it converges. As in the calculations of Sec. 4, it is found necessary to cut off the absorptive parts A_1, A_2 and A_3 at high energies, before calculating the singularities of the low angular momentum states in the next iteration. However, the cutoff is only applied above the threshold for processes neglected in the unitarity condition, and in particular, above the threshold for pair production in the reaction I.

This approximation could be regarded as the first of a series of approximations in which more and more of the contributions to the spectral functions are included, until we ultimately reach a solution in which the unitarity condition in the one-meson approximation is satisfied for every angular momentum state. In the higher approximations the spectral functions are no longer determined by perturbation theory, but, once the contribution from the crossing term enters, they will have to be recalculated after each iteration. However, it would be more worthwhile to go beyond the one-meson approximation at the same time as we took the higher contributions to the spectral functions into account. In other words, we continue to put the reactions I, II, and III on the same footing, bringing in the higher intermediate states of all three together. If the approximation scheme converged, the exact unitarity condition of the three reactions would finally be satisfied for all angular momentum states. Needless to say, one would not in practice be able to go beyond the first one or two approximations.

The number of angular momentum states for which the unitarity condition is applied separately will, as has been explained in the last section, depend on the behavior of A and B as t (or s_c) tends to infinity with s constant. However, in our first approximation, it should be sufficient to treat separately only states with $j=\frac{1}{2}$ and $j=\frac{3}{2}$, as the other angular momentum states will not be important below the threshold for pion production. If we went beyond the one-meson approximation we would probably have to treat some higher angular momentum states separately in any case, since, for instance, two pions both in a (3,3) resonance state with a nucleon could form a $D_{\frac{5}{2}}$ state. For reaction III, one would have to treat separately S states and possibly P states as well.

If one neglected the nucleon-antinucleon intermediate state in the reaction III and only took the two-pion intermediate state into account, all three spectral functions A_{13}, A_{23}, and A_{12} would be zero, since they all begin above the threshold for processes which are being neglected. The entire scattering amplitude would then consist of "subtraction terms" for one or other of the dispersion relations. This may be the best first approximation from the point of view of the amount of work required and the accuracy of the result, as the nucleon-antinucleon intermediate state is a good deal heavier than multipion states which are being neglected. Though the spectral functions are not now brought in at all, it will of course be realized that the only justification for the approximation is that it is the first of a series of approximations which do involve the spectral functions. In this approximation, if the crossing term is neglected in the calculation of the pion-pion scattering amplitude, only intermediate S states occur in reaction III, so that the unitarity condition for the P states will not enter.

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EQUALITY OF THE NUCLEON AND ANTINUCLEON TOTAL INTERACTION CROSS SECTION AT HIGH ENERGIES

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The dispersion relations are used to show that the total interaction cross sections are equal for particles and antiparticles at high energies.

FROM the dispersion relations for elastic scattering of nucleons through angle zero[1-3] one can establish that the total interaction cross sections for nucleons and antinucleons must be equal at sufficiently high energies.

To be specific, let us consider scattering of protons and antiprotons through the angle zero. The dispersion relations for the scattered amplitudies, averaged over spins, are of the form

$$D_+(E) = \frac{1}{2}\left(1+\frac{E}{M}\right)D_+(M) + \frac{1}{2}\left(1-\frac{E}{M}\right)D_-(M) + \frac{p^2}{4\pi^3}\int_M^\infty \frac{dE'}{p'}\left[\frac{\sigma_+(E')}{E'-E} + \frac{\sigma_-(E')}{E'+E}\right] + \frac{f^2}{\mu^2}\,\frac{p^2}{M-\mu^2/2M-E} + \frac{p^3}{4\pi^3}\int_0^M \frac{idE'}{p'}\left[\frac{\sigma_+(E')}{E'-E} + \frac{\sigma_-(E')}{E'+E}\right]; \tag{1}$$

$$D_-(E) = \frac{1}{2}\left(1+\frac{E}{M}\right)D_-(M) + \frac{1}{2}\left(1-\frac{E}{M}\right)D_+(M) + \frac{p^3}{4\pi^3}\int_M^\infty \frac{dE'}{p'}\left[\frac{\sigma_-(E')}{E'-E} + \frac{\sigma_+(E')}{E'+E}\right] + \frac{f^2}{\mu^2}\,\frac{p^3}{M-\mu^3/2M+E} + \frac{p^3}{4\pi^3}\int_0^M \frac{idE'}{p'}\left[\frac{\sigma_+(E')}{E'+E} + \frac{\sigma_-(E')}{E'-E}\right]. \tag{1'}$$

Here $D_+(E)$ is the real part of the elastic scattered amplitude, averaged over spins, for a proton of energy E scattered by a proton at rest, $D_-(E)$ is the real part of the elastic scattered amplitude for an antiproton of energy E scattered by a proton at rest, $\sigma_+(E)$ is the total scattering cross section for a proton with energy E, and $\sigma_-(E)$ is the total scattering cross section for an antiproton with energy E. For energies from zero to M, the function $\sigma_+(E)$ is the analytic continuation of $\sigma_+(E)$ for energies $E > M$, and $\sigma_-(E)$ is the analytic continuation of $\sigma_-(E)$ into the "nonphysical" region $E < M$.

As the energy E approaches infinity, $\sigma_+(E)$ and $\sigma_-(E)$ approach the constant values $\sigma_+(\infty)$ and $\sigma_-(\infty)$. This follows simply from the fact that all strong interactions approach zero exponentially for large values of the impact parameter ρ (here the form of the factor multiplying the exponential is entirely insignificant). Had we wished to consider the electromagnetic interaction of a proton or antiproton with a proton, we could have taken into account the screening of the target proton charge at large distances by the atomic electrons, and also obtained a constant total cross section at sufficiently high energies. It is true that in this way a constant cross section would be obtained only at energies of the order of $E \sim M^2/me^2 \sim 10^{14}$ ev. If, however, we were not to take into account the weak electromagnetic interaction, then up to a quantity of order $e^2 \ln(E/M)$ the total cross section would reach a constant value even at an energy of the order of 10^{10} ev. Allowing E to approach infinity and maintaining only the largest terms, Eqs. (1) and (1′) lead to

$$D_+(E) = \frac{E}{2M}[D_+(M) - D_-(M)] - \frac{f^2}{\mu^2}E + \frac{E}{4\pi^2}\int_0^M \frac{idE'}{p'}[\sigma_-(E') - \sigma_+(E')] + \frac{p^2}{4\pi^2}\int_M^\infty \frac{dE'}{p'}\left[\frac{\sigma_+(E')}{E'-E} + \frac{\sigma_-(E')}{E'+E}\right]; \tag{2}$$

$$D_-(E) = \frac{E}{2M}[D_-(M) - D_+(M)] + \frac{f^2E}{\mu^2} + \frac{E}{4\pi^2}\int_0^M \frac{idE'}{p'}[\sigma_+(E') - \sigma_-(E')] + \frac{p^2}{4\pi^2}\int_M^\infty \frac{dE'}{p'}\left[\frac{\sigma_-(E')}{E'-E} + \frac{\sigma_+(E')}{E'+E}\right]. \tag{2'}$$

In these equations let us first consider the integral from M to ∞. Starting at a certain energy ϵ, both cross sections σ_+ and σ_- may be considered constant and equal to $\sigma_+(\infty)$ and

$\sigma_-(\infty)$. We may therefore write $(E \gg \epsilon)$*

$$\frac{E^2}{4\pi^2}\int_M^\infty \left[\frac{\sigma_+(E')}{E'-E}+\frac{\sigma_-(E')}{E'+E}\right] dE' \frac{1}{\sqrt{E'^2-M^2}}$$

$$= \frac{E^2}{4\pi^2}\int_M^\epsilon \left[\frac{\sigma_+(E')}{E'-E}+\frac{\sigma_-(E')}{E'+E}\right]\frac{dE'}{p'} \tag{3}$$

$$+\frac{E^2}{4\pi^2}\int_\epsilon^\infty \frac{dE'}{E'}\left[\frac{\sigma_+(\infty)}{E'-E}+\frac{\sigma_-(\infty)}{E'+E}\right] = \frac{E}{4\pi^2}\int_M^\epsilon [\sigma_-(E')-\sigma_+(E')]\frac{dE'}{p'}$$

$$+\frac{E^2}{4\pi^2}\int_\epsilon^\infty \frac{dE'}{E'}\left[\frac{\sigma_+(\infty)}{E'-E}+\frac{\sigma_-(\infty)}{E'+E}\right].$$

When $E \gg \epsilon$, the asymptotic value of (3) is

$$\frac{E}{4\pi^2}\left(\ln\frac{E}{\epsilon}\right)[\sigma_-(\infty)-\sigma_+(\infty)] \tag{4}$$

[where we have assumed that $\sigma_-(\infty) \neq \sigma_+(\infty)$]. The integral from M to ϵ is, with our conditions, proportional to E:

$$\frac{E}{4\pi^2}\int_M^\epsilon [\sigma_-(E')-\sigma_+(E')]\frac{dE'}{p'}. \tag{5}$$

By combining (3), (4), and (5), Eqs. (2) and (2′) become

$$D_+(E) = E\left\{\frac{D_+(M)-D_-(M)}{2M}-\frac{f^2}{\mu^2}\right.$$

$$+\frac{1}{4\pi^2}\int_0^M \frac{i\,dE'}{p'}[\sigma_-(E')-\sigma_+(E')] \tag{6}$$

$$\left.+\frac{1}{4\pi^2}\int_M^\epsilon \frac{dE'}{p'}[\sigma_-(E')-\sigma_+(E')]+\frac{1}{4\pi^2}[\sigma_-(\infty)-\sigma_+(\infty)]\ln\frac{E}{\epsilon}\right\},$$

$$D_-(E) = E\left\{\frac{D_-(M)-D_+(M)}{2M}+\frac{f^2}{\mu^2}\right.$$

$$+\frac{1}{4\pi^2}\int_0^M \frac{i\,dE'}{p'}[\sigma_+(E')-\sigma_-(E')] \tag{6′}$$

$$\left.+\frac{1}{4\pi^2}\int_M^\epsilon \frac{dE'}{p'}[\sigma_+(E')-\sigma_-(E')]+\frac{1}{4\pi^2}[\sigma_+(\infty)-\sigma_-(\infty)]\ln\frac{E}{\epsilon}\right\}.$$

We see that if $\sigma_+(\infty) \neq \sigma_-(\infty)$, the principal term which determines $D_+(E)$ and $D_-(E)$ increases faster than E, and is proportional to $E \times \ln(E/\epsilon)$. This result contradicts the conclusion that the interaction decays exponentially for large distances. Let us write out the general expression for the elastic scattered amplitude $A(0)$ for the angle zero. For simplicity, we shall not take account of spin, noting that the orbital quantum number l is very large at high energies, so that the replacement of l by $l \pm 1$, which must be made in order to account for spin, will change nothing in the following considerations.

At high energies, when $l \gg 1$, we have

$$A(0) = \frac{1}{2E}\sum_l (e^{2i\eta_l}-1)\,l, \tag{7}$$

where η_l is the phase of the wave with orbital angular momentum l. For large l we may use the semiclassical impact parameter l/E. When l/E is much larger than the radius of the interaction ρ (which is of the order of the Compton wavelength of the π meson), η_l decays exponentially with l. It follows from this that the effective upper limit in Eq. (7) is a quantity of order $E\rho$. (This result is independent of the form of the factor multiplying the exponential of the interaction.)

Since the modulus of $e^{2i\eta_l}-1$ is no greater than 2, the order of magnitude of the modulus of $A(0)$ is

$$|A(0)| \leqslant CE\rho^2, \quad C \sim 1. \tag{8}$$

It follows from this that $D_\pm(E)$ cannot contain terms proportional to $[\sigma_+(\infty)-\sigma_-(\infty)]E\ln(E/\epsilon)$. Therefore*

$$\sigma_+(\infty) = \sigma_-(\infty). \tag{9}$$

Thus at large energies the total antiproton and proton cross sections are equal. When we recall that at energies of several hundreds of Mev these cross sections are very different (σ_- is much greater[4,5] than σ_+), it becomes evident that at large energies we should expect one or both of these cross sections to be highly energy dependent.

An equation similar to (9) should hold also for the asymptotic values of the total cross sections $\sigma_n(\infty)$ and $\sigma_{\bar{n}}(\infty)$ for neutrons and antineutrons on protons, namely

$$\sigma_{\bar{n}}(\infty) = \sigma_n(\infty). \tag{10}$$

From statistical considerations and the requirement of isotopic invariance,[6] we may establish the equality of $\sigma_n(\infty)$ and $\sigma_+(\infty)$, as well as that of $\sigma_{\bar{n}}(\infty)$ and $\sigma_-(\infty)$. At high energies, therefore, the proton, the neutron, the antiproton, and the antineutron should all have the same limiting values for their cross sections.

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*Let us bear in mind that we take the principal part of the integral at the point $E' = E$. The same is true with respect to the singularity at $E = M$ (where σ_- approaches ∞ as $1/v$, with v being the velocity). In treating this singularity, one must treat simultaneously the intervals from zero to M and from M to ϵ.

*Equation (9) has been derived in a different way by B. V. Medvedev and D. V. Shirkov.

Similar considerations hold also for K mesons. This means that at sufficiently high energies (of the order of 10^{10} ev, if electromagnetic interactions are not taken into account) the K^+, K^-, K^0, and $\overline{K}^0$ mesons should all have the same total cross section. Similarly, the π^+ and π^- cross sections should also approach equality as $E \to \infty$.[7] If we apply our result to hyperons, we obtain the following.

(1) The Λ and anti-Λ (i.e., the $\overline{\Lambda}$) have the same cross section $\sigma_\Lambda(\infty)$.

(2) The Σ^+, Σ^-, Σ^0, $\overline{\Sigma}^+$, $\overline{\Sigma}^-$, and $\overline{\Sigma}^0$ cross sections approach the common value $\sigma_\Sigma(\infty)$ as $E \to \infty$.

(3) The Ξ^-, Ξ^0, $\overline{\Xi}^-$, $\overline{\Xi}^0$ have the same cross section $\sigma_\Xi(\infty)$.

We note also that from (2), (2′), and (9) it follows that at large energies the main contributions to $D_+(E)$ and $D_-(E)$ are proportional to E and differ only in sign. This together with (9) shows that the differential cross sections for elastic scattering of nucleons and antinucleons by nucleons through the angle zero approach equality as $E \to \infty$.

In conclusion, I should like to thank N. N. Bogoliubov, B. L. Ioffe, L. D. Landau, B. V. Medvedev, D. V. Shirkov, and I. M. Shmushkevich for interesting discussions concerning this work.

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Problem of Proving the Mandelstam Representation*

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It is shown that sufficiency conditions for the validity of the Mandelstam representation of a collision amplitude can be stated in terms of the location of singularities of its physical branch when one of the energy variables is real and positive and the second independent energy variable is complex. It is also shown that in perturbation theory the real singularities of the physical branch must correspond to positive Feynman parameters provided that none of the curves of singularities identified in this way have turning points in positive spectral regions. The same condition will exclude complex singularities in the physical sheet that are connected to the curves of singularities on its real boundary. If there are any disconnected complex singularities they must correspond to complex Feynman parameters.

It is concluded that sufficiency conditions for the Mandelstam representation are the absence of turning points in positive spectral regions and the absence of disconnected complex singularities in the physical sheet. In the equal-mass case these conditions are both necessary and sufficient, and real singularities in the physical sheet can be identified by requiring that the Feynman parameters shall be positive. The conditions are satisfied by ladder diagrams when there are no anomalous thresholds. [It has now been proved that the representation is correct for every order in perturbation theory. See Phys. Rev. Letters 5, 213 (September 1, 1960), letter by the author. The proof requires some of the results of this paper.]

I. INTRODUCTION

THE aim of this paper is to obtain sufficiency conditions for the validity of the Mandelstam representation[1] in a form that is convenient for study in perturbation theory. The Mandelstam representation implies that a scattering amplitude has a branch that is analytic in the complex planes of the invariant energies, s, t, and u except for branch cuts and poles on the positive real axes. These are defined by the equations

$$s=(p_1+p_2)^2,\quad t=(p_1+p_4)^2,\quad u=(p_2+p_3)^2, \tag{1.1}$$

and

$$\sum_1^4 p_i=0,\quad p_i^2=M_i^2,\quad s+t+u=\sum_1^4 M_i^2, \tag{1.2}$$

where p_i are the external four-momenta of the colliding particles. In Sec. 2, it is shown that sufficiency conditions for the representation can be stated in terms of the location of singularities of the physical branch of the amplitude $A(s,t,u)$ when one of the variables s, t, u is real and positive.

In order to study whether these conditions are in fact satisfied by an amplitude defined by a perturbation solution in quantum field theory, we must have a method for locating singularities of the physical branch without considering also those of the more complicated unphysical branches. The physical branch is defined so that it satisfies the causality requirements of quantum field theory in the physical scattering regions of the real, s, t, u plane which forms the boundary of the physical sheets of the complex planes. This definition leads to the usual prescription of associating every internal mass with a small negative imaginary part to give $(m^2-i\epsilon)$ in the Feynman integrals of the perturbation series. Singularities of these integrals can always be associated with critical values of the Feynman parameters in their integrands. It is shown in Sec. 3 that under certain conditions the real singularities of the physical branch can be identified by the requirement that the critical values of the Feynman parameters must be positive. The conditions used are the absence of turning points in those curves of singularities for which the Feynman parameters are positive. These conditions are sufficient, but probably are not always necessary in the general mass case. They also ensure the absence of complex singularities that connect to real curves of singularities on the boundary of the physical sheet. These conclusions are not affected by the possible existence of disconnected complex singularities. If these do exist, the Mandelstam representation would have to be modified by the addition of complex contours of integration.

The restriction to positive Feynman parameters greatly simplifies the discussion of curves of singularities. The problem of determining whether an amplitude given by a perturbation series satisfies these conditions has been discussed elsewhere.[2] Some results of that paper (hereinafter denoted as I) will be used in Sec. 3.

II. SUFFICIENCY CONDITIONS

In this section we obtain conditions on the location of the singularities of a branch of a collision amplitude for it to satisfy the Mandelstam representation. The representation will be established by a double application of Cauchy's theorem by making explicit assumptions about the location of singularities. These assumptions then provide the sufficiency conditions that we require.

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For simplicity we use the equal-mass problem to illustrate our assumptions. The general mass case requires special consideration in the problem of proving these assumptions, but in establishing their sufficiency it has no special significance.

Assumption 2A

There is a region of the real s, t plane and a complex neighborhood of this region in which the amplitude $A(s,t)$ has no singularities or branch cuts.

Assumption 2B

With t real and within the bounds set by the analytic region of assumption 2A, the amplitude $A(s,t)$ has no singularities in the complex s plane except for branch points or poles on the real axis.

We will not explicitly include poles in our formulas, since they give only trivial changes. The region of analyticity in the equal-mass case has been obtained in I. A single application of Cauchy's theorem in the complex s plane gives,

$$A(s,t)=\frac{1}{2\pi i}\int_{4m^2}^{\infty}\frac{[A(s'+i\epsilon,t)-A(s'-i\epsilon,t)]}{(s'-s)}ds'$$
$$-\frac{1}{2\pi i}\int_{-t}^{-\infty}\frac{[A(s'+i\epsilon,t)-A(s'-i\epsilon,t)]}{(s'-s)}ds'. \quad (2.1)$$

The region of analyticity and the two integration contours are illustrated in Fig. 1, where the contours are denoted (a) and (b).

Assumption 2C

For all real and positive values of s except for a discrete set of points, the amplitude $A(s,t)$ has no singularities in the complex t plane except for branch points on the real axis in the ranges $4m^2\leq t$, and $t\leq -s$.

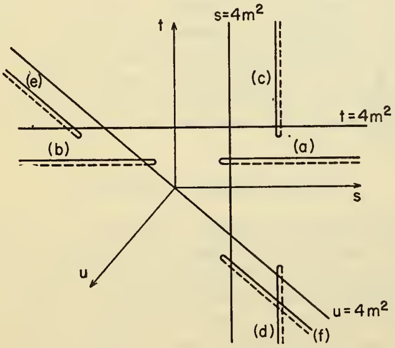

FIG. 1. The real s, t, u plane ($u=4m^2-s-t$). Branch points and contours of integration in the complex planes of s, t, u are indicated.

We can allow ϵ in Eq. (2.1) to tend to zero except in a negligible part of the contours of integration. The assumption 2C permits a second application of Cauchy's theorem, which transforms the first integral on the right of Eq. (2.1) into a double-dispersion integral. The appropriate contours are denoted (c) and (d) in Fig. 1, and the double integral is

$$\frac{1}{(2\pi i)^2}\int_{4m^2}^{\infty}ds'\int_{4m^2}^{\infty}dt'\frac{\rho_1(s',t')}{(s'-s)(t'-t)}$$
$$+\frac{1}{(2\pi i)^2}\int_{4m^2}^{\infty}du'\int_{m^2}^{\infty}ds'\frac{\rho_2(u',s')}{(u'-u)(s'-s)}, \quad (2.2)$$

where

$$\rho_1(s',t')=\underset{\epsilon\to 0}{\text{limit}}[A(s'+i\epsilon,t'+i\epsilon)-A(s'-i\epsilon,t'+i\epsilon)$$
$$-A(s'+i\epsilon,t'-i\epsilon)+A(s'-i\epsilon,t'-i\epsilon)] \quad (2.3)$$

and $\rho_2(u',s')$ is defined from Eq. (2.3) by the substitution

$$t'=4m^2-s'-u'. \quad (2.4)$$

If we made an assumption similar to assumption 2C but for s real and negative, the integrand of the second integral on the right of Eq. (2.1) could be transformed by Cauchy's theorem, and we could formally obtain a double integral. However this would not be in a useful form. This is clear from Fig. 1, which shows that the values of t in the integrand that we wish to transform lie on the cut in the complex t plane. Hence there would be an essential contribution to the integral from the neighborhood of these values.

A useful transformation of the integrand of the second integral in Eq. (2.1) can be obtained by using oblique axes. These are defined by a contour along which u is fixed and real. We require a further assumption.

Assumption 2D

For all real and positive values of u except for a discrete set of points, the amplitude $A(4m^2-t-u,u)$ has no singularities in the complex t plane except for branch points on the real axis in the ranges $4m^2\leq t$, and $t\leq -u$.

With this assumption, the integrand of the second integral in Eq. (2.1) can be transformed into a dispersion integral along the contours (e) and (f) in Fig. 1. This leads to double-dispersion integrals

$$-\frac{1}{(2\pi i)^2}\int_{-t}^{-\infty}ds_1\int_{4m^2}^{\infty}dt_2\frac{\rho_3(t',u')}{(t-t)(u'-u)}$$
$$-\frac{1}{(2\pi i)^2}\int_{-t}^{-\infty}ds_1\int_{4m^2}^{\infty}ds_2\frac{\rho_2(u',s')}{(u'-u)(s'-s)}. \quad (2.5)$$

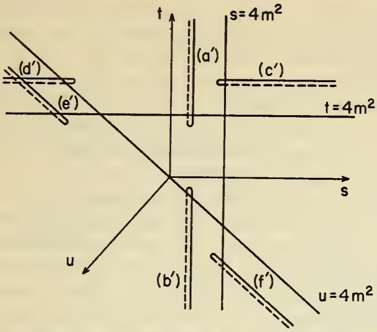

FIG. 2. The real s, t, u plane showing contours used in the second double application of Cauchy's theorem.

The integration elements ds_1 are taken with t' fixed, and dt_2 and ds_2 are taken with u' fixed. The function $\rho_3(t',u')$ is obtained from Eq. (2.3) by using Eq. (2.4).

By repeating the above procedure with the first transformation made in the complex t plane, we obtain integrals along the paths (a'), (b'), (c'), (d'), (e'), and (f') in Fig. 2. Clearly the double integrals are over the same regions as those in expressions (2.2) and (2.5). An additional assumption has to be made.

Assumption 2E

For all real and positive values of t except for a discrete set of points, the amplitude $A(s,t)$ has no singularities in the complex s plane except for branch points on the real axis in the ranges $4m^2\leq s$, and $s\leq -t$.

The integrals over oblique axes in this second case are the same as in expression (2.5), but the integration elements are now dt_1 taken with s' fixed, and dt_2 and ds_2 taken with u' fixed as before. Using Eq. (2.4), we have

$$du'=-ds'-dt'=-ds_1-dt_1. \qquad (2.6)$$

This permits us to combine the double integrals. After dividing by two, we obtain the Mandelstam representation for $A(s,t)$, namely, the two terms shown in expression (2.2) and the term

$$\frac{1}{(2\pi i)^2}\int_{4m^2}^{\infty}dt'\int_{4m^2}^{\infty}du'\frac{\rho_3(t',u')}{(t'-t)(u'-u)}. \qquad (2.7)$$

For the complete amplitude, ρ_1, ρ_2, and ρ_3 will be the same functions in the equal-mass case. For parts of the amplitude, such as terms in the perturbation series, they will not in general be the same functions.

We conclude that sufficiency conditions for the Mandelstam representation require (a) a real region and its complex neighborhood in which a branch of the amplitude $A(s,t,u)$ is free from singularities, and (b) analyticity when each variable in turn is real and positive and the others are complex but satisfy Eq. (1.2).

III. THE PHYSICAL BRANCH OF A COLLISION AMPLITUDE

The Mandelstam representation assumes it is the physical branch of a collision amplitude that has the simple analyticity properties that were discussed in Sec. 2. In this section we show that the definition of the physical branch in perturbation theory in terms of the Feynman amplitude in the real scattering regions sufficient under certain conditions to define the amplitude unambiguously throughout the physical sheet. It is shown further that real singularities of the physical branch under the same conditions can be identified by requiring critical values of Feynman parameters to be real and positive.

A. The Real Boundary of the Physical Sheet

A scattering amplitude satisfying causality conditions and giving the correct scattering in physical parts of the real s, t plane can be expressed as a sum of terms corresponding to the Feynman diagrams for the perturbation series. Each term has the form

$$F(s,t)=\operatorname*{limit}_{\epsilon\to 0}c\int dk_1\cdots\int dk_l\prod_{i=1}^{n}\frac{1}{(q_i^2-m_i^2+i\epsilon)} \qquad (3.1)$$

in which q_i is the four-momentum in an internal line and depends linearly on the internal momenta k_j and the external momenta p_k.

The integral on the right of Eq. (3.1) can be transformed, giving

$$F_\epsilon(s,t)=c_1\int dk_1\cdots\int dk_l\int_0^1 d\alpha_1\cdots \times\int_0^1 d\alpha_n\frac{\delta(1-\sum\alpha_i)}{[\psi(k,\alpha,p)]^{n+1}}, \qquad (3.2)$$

where

$$\psi(k,\alpha,p)=\sum_{i=1}^{n}\alpha_i(q_i^2-m_i^2+i\epsilon). \qquad (3.3)$$

In the physical scattering regions, each four-momentum q_i is a real Minkowski vector. Hence for $\epsilon>0$ we have

$$q_i^2-m_i^2+i\epsilon\neq 0. \qquad (3.4)$$

This proves that for $\epsilon>0$ in the physical scattering regions, $F_\epsilon(s,t)$ is nonsingular. It is known also that as ϵ tends to zero in these regions the amplitude becomes singular at normal thresholds, and its component F may become singular at these points. The amplitude may also have singularities below the first normal threshold if certain mass inequalities hold. These singularities give anomalous thresholds. If there are no anomalous thresholds, the Euclidean region of the real s, real t plane is free from singularities (see I, Sec. 4 and 8).

The real s, real t plane is defined as the boundary of the physical sheets in s, t, and u by taking branch cuts in the complex planes of these variables along their real axes from the leading real branch point to infinity. In order to make this definition meaningful, we must show that the physical branch of the amplitude can be defined near the real boundary of the physical sheet.

B. Near the Real Boundary of the Physical Sheet

When ϵ is positive, the integration in Eq. (3.2) over the internal momenta can be carried out unambiguously. This gives

$$F_\epsilon(s,t)=c_2\int_0^1 d\alpha_1\cdots\int_0^1 d\alpha_n\frac{\delta(1-\sum\alpha_i)C(\alpha)^{n-2l-1}}{D_\epsilon(\alpha,s,t)^{n-2l}}, \quad (3.5)$$

where D_ϵ is the discriminant of ψ as a function of k, and C is the discriminant of ψ_0 obtained from ψ by putting $m_i=0$, $s=0$, and $t=0$. The discriminant D_ϵ has the form

$$D_\epsilon(\alpha,s,t)=sf(\alpha)+tg(\alpha)-K_\epsilon(\alpha), \quad (3.6)$$

where $K_\epsilon(\alpha)$ is a function of the external masses M_r^2, the internal masses $m_i^2-i\epsilon$, and the variables α. The coefficient of $i\epsilon$ is $C(\alpha)$, which is positive when the variables α are real and positive since it is then the discriminant of a positive definite quadratic form. It follows that with $\epsilon>0$, and s and t both real, we have

$$D_\epsilon(\alpha,s,t)\neq 0 \quad (3.7)$$

along the path of integration in the α variables. This result shows that analytic continuation through the whole of the real s, real t plane can be carried out for $F_\epsilon(s,t)$.

For certain real values of s and t, $F_\epsilon(s,t)$ becomes singular as ϵ goes to zero. At these values the singularities of the integrand of Eq. (3.5) either pinch the contour of integration or occur at the end point, $\alpha_i=0$. These singularities are zeros of $D(\alpha,s,t)$ as a function of the α variables. They must also occur at real positive values of each α_i, since, when ϵ is positive, Eq. (3.7) holds and it is never necessary to distort the contour of integration from the real path, 0 to 1.

If there are no turning points in the curves of singularities, $f(\alpha)$ and $g(\alpha)$ are both positive at the critical α values that correspond to a singular point of the amplitude (see I, particularly Sec. 7). We will assume for this discussion that this condition is valid. Then near the critical values of α we can write

$$D_\epsilon(\alpha,s,t)=(s+i\epsilon')f(\alpha)+(t+i\epsilon'')g(\alpha)-K(\alpha) \quad (3.8)$$

$$=D(\alpha,\, s+i\epsilon',\, t+i\epsilon'') \quad (3.9)$$

where ϵ' and ϵ'' are positive, and

$$K(\alpha)=K_{(\epsilon=0)}(\alpha). \quad (3.10)$$

The relation (3.8) is not valid in general at all positive values of α, since f and g do not always remain positive away from the critical values of α. However, away from these values, at least one of the α_i will give singularities that are not coincident, and they can therefore be avoided by a suitable small distortion of the contour of integration. This distortion is not required near the critical values of α, where the imaginary part of s or t suffices to prevent the singularities being on the contour of integration.

At a normal threshold in s, for example, $g(\alpha)$ is zero. Then the singularity can be avoided by giving s a small positive imaginary part. The above discussion of curves of singularities shows that, in the absence of turning points, a small positive imaginary part in s will also avoid these singularities in the region where s is positive. For a discrete set of real values of t (the normal thresholds), the location of singularities is independent of s, but analytic continuation past them is defined by giving t a small positive imaginary part. The discussion can be extended in an obvious manner to regions where the variable u is positive. These results are summarized in the following theorem.

Theorem 3A

Provided there are no turning points in the curves of singularities of the physical branch of the amplitude in positive spectral regions, the amplitude can be analytically continued through the complex neighborhood of the real boundary of the physical sheet.

C. Positive Feynman Parameters and the Physical Branch of the Amplitude

We consider first the situation in which the curves of singularities have no turning points. From Theorem 3A the amplitude is single-valued under analytic continuation near the entire real boundary of the physical sheet. In the limit as the real boundary is approached there may be singularities. These singularities must correspond to positive values of the Feynman parameters α, since the path of integration does not need to be distorted in any significant way in the complex neighborhood of the boundary. It follows that negative Feynman parameters are important on or near the real boundary only if the amplitude is not single-valued on the physical sheet. It can be multivalued only if there are complex singularities. From Theorem 3A these cannot be connected to singularities on the real boundary. Hence they must be disconnected complex singularities. Even if these disconnected singularities exist, we can still choose that branch of the function whose singularities are given by the positive α condition and make it single-valued by introducing suitable branch cuts in the complex part of the physical sheet. This will be the physical branch of the function.

Theorem 3B

Provided there are no turning points in curves of singularities obtained with the positive condition on the Feynman parameters, the physical branch of the amplitude can be defined so that (a) it is single-valued in the physical sheet, and (b) all its singularities on the real boundary of the physical sheet are given by the positive condition on the Feynman parameters.

This discussion assumes the absence of turning points. If there are spurious turning points (see I, Sec. 7D), there will be complex branch cuts which prevent the simple analytic continuation obtained in Theorem 3A. With anomalous turning points, there are two possibilities. Either the resulting complex singularities go into a nonphysical sheet, in which case Theorem 3A will still be valid, or they remain in the physical sheet and connect to real singularities on another branch of the curve of singularities. In the latter case, the conditions of Theorem 3A do not hold, and Feynman parameters may not always be positive for the physical branch of the amplitude. Barmawi has pointed out that this happens in fourth-order perturbation theory with superanomalous thresholds and the physical branch has some singularities corresponding to negative Feynman parameters.[3]

In the equal-mass case, there are no anomalous thresholds. Hence any turning points that exist must be spurious, and if they occur in positive spectral regions, they will cause a breakdown of the Mandelstam representation. Hence in this case the absence of turning points in positive spectral regions is also a necessary condition for the validity of the representation.

D. Disconnected Complex Singularities

When s is real and t is complex with a positive imaginary part, we can see from Eq. (3.8) that $D(\alpha,s,t)$ is nonzero provided α is real and $g(\alpha)$ is nonzero. Thus the zeros of D do not cross the real α axis except where $g(\alpha)$ is zero. In the simple case of ladder diagrams, or diagrams formed by reducing ladders, the functions $f(\alpha)$ and $g(\alpha)$ consist only of positive terms on the path of integration. Then unless there are anomalous thresholds, D will not vanish for s real and t complex, and there will be no disconnected complex singularities. For the equal-mass case, ladder diagrams have no anomalous thresholds and hence no disconnected singularities, nor do they have spurious turning points. It follows that these ladder diagrams satisfy the Mandelstam representation. This result has also been proved by Cutkowski[4] and by Wanders[5] using the Bethe-Salpeter equation.

In general, in a region where $g(\alpha)$ is negative, some of the zeros of D will cross the real α axis as t goes complex, and will lead to distortion of the contour of integration. When t is nearly real, this distortion is unimportant, as was seen in Theorem 3A. As the imaginary part of t increases, the variables α become complex and the condition that $g(\alpha)$ vanishes is no longer necessary for other zeros of D to cross the real α axis. There may then be new coincident singularities of the integrand on the right of Eq. (3.5). It is still necessary for D as well as its derivatives with respect to each α_i to be zero if the integral in Eq. (3.5) is also to be singular. The latter condition cannot be satisfied unless some of the α_i are complex.

If there are such disconnected complex singularities, they will cause a breakdown of the Mandelstam representation by the introduction of additional branch cuts in the physical sheet. The difficulty in eliminating the possibility that they exist lies in their very tenuous relation to the positive condition on Feynman parameters, which holds only for real singularities under the conditions discussed earlier. This obscures the identification of the Riemann sheets in which disconnected complex singularities occur, since in all sheets including the physical one, they involve Feynman parameters that are complex and have no simple relation to their values for real singularities.

Note added in proof. It has now been proved that every term in the perturbation expansion of the scattering amplitude satisfies the Mandelstam representation. The proof will be described in a paper by the author to be submitted to the Physical Review. The proof was outlined at the Rochester Conference 1960, and in Phys. Rev. Letters 5, 213 (September 1, 1960).

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[3] M. Barmawi, Physics Department, University of Chicago (private communication).

[4] R. Cutkowski, Physics Department, Carnegie Institute of Technology (private communication).

[5] G. Wanders, Institut für Theoretische Physik der Universität, Hamburg, 1960 (unpublished).

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The Application of Dispersion Relations in Quantum Field Theory[1]

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I. Introduction

In recent years much effort has gone into the study of implications of those assumptions of quantum field theory which are model-independent, i. e. which do not depend on the particular form of the equations of motion. It has been found that Lorentz invariance, local commutativity („microscopic causality") and simple considerations of the spectrum of the energy operator are sufficient to establish as generally true a number of properties which had earlier been discovered in perturbation treatments of particular interactions. From a practical point of view, the most useful of these have been the analyticity properties of matrix elements of the S-matrix, because the expression of these in terms of dispersion relations has in some cases led to relations between experimentally measurable quantities. The agreement of these relations with experiment has been encouraging, and the very weak „model-dependence" of the dispersion relations has led to the hope that by the use of the correct analyticity and unitarity properties one may go a long way towards fixing the forms of the scattering amplitudes. This hope has been responsible for a new approach to approximation methods in quantum field theory: the emphasis is now on incorporating as many of the model-independent features as possible into a general representation of the scattering amplitude. This is to be contrasted with earlier attempts to mould the approximate equations of motion after the form of the Hamiltonian postulated to represent the interaction. The new methods have the advantage that no renormalization procedure is necessary — an important advantage since some of the earlier approximation methods could not be renormalized — and they appear to be more convincing than the selective summing up of patterns of Feynman graphs.

This article provides a fairly detailed, if selective, summary of the assumptions made, and the methods used in most recent approaches to the meson-nucleon problem. In sec. II the expression of S-matrix elements in terms of operators is reviewed; in sec. III the relation between the S-matrix elements and the scattering amplitude is obtained, and some of the complications coming from spin and i-spin are discussed (see also appendices A and B). In sec. IV we discuss the unitarity of the S-matrix and its field-theoretic generalization. In sec. V the analyticity properties of scattering amplitudes are discussed, though not proved, and in sec. VI the combination of these analyticity properties with the optical theorem in the forward scattering dispersion relations is discussed, in particular in relation to the high energy theorem of Pomeranchuk. In sec. VII the Mandelstam representation for the scattering amplitude is introduced and its properties are discussed in great detail; its use in approximation methods is illustrated in sec. VIII where the formulation of the pion-pion scattering problem is discussed, and in sec. IX where the more complicated pion-nucleon problem is briefly considered.

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[1] Based on lectures delivered at the Max-Planck Institute for Physics and Astrophysics, Munich, in the winter 1959—60.

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II. The Scattering Matrix

The basic quantities in the quantum theory of fields are local field operators. These are classified according to their transformation properties under the inhomogeneous Lorentz group, under reflections, and perhaps under certain other "internal symmetry" groups. These operators are defined in a Hilbert space, and they are assumed to form an irreducible ring, i. e. all vectors in the space can be constructed out of polynomials of these operators acting on any one of the vectors. Another assumption made about the local field operators is that they satisfy the local commutativity conditions, i. e. that

$$\left.\begin{array}{l} [\varphi_\alpha(x),\ \varphi_\beta(y)] = 0 \\ [\varphi_\alpha(x),\ \psi(y)] \ \ = 0 \\ \{\psi(x),\ \overline{\psi}(y)\} \ \ = 0 \\ \ldots\ldots\ldots \text{ etc.} \end{array}\right\} \text{ for } (x-y)^2 \equiv (\mathbf{x}-\mathbf{y})^2 - (x_0-y_0)^2 > 0.$$

The Hilbert space is taken to be a direct sum of vectors representing a vacuum state $|0\rangle$, all discrete (one-particle) states, and appropriately symmetrized two-, three-, ... particle states. Among the one particle states are not only the so-called "elementary particles", but all stable states such as the deuteron, for example.

In order to represent the Hilbert space, we define, following ZIMMERMANN[1]), the operators

$$\varphi_{\substack{\text{out}\\ \text{in}}}(x) = \varphi(x) + \int dx'\, \Delta_{\substack{\text{adv.}\\ \text{ret.}}}(x-x';m)\,(\Box' - m^2)\,\varphi(x') \tag{2.1}$$

corresponding to spin zero fields $\varphi(x)$. Similarly for the spin $^1/_2$ fields $\psi(x)$, $\overline{\psi}(x)$, we write

$$\psi_{\substack{\text{out}\\ \text{in}}}(x) = \psi(x) + \int dx'\, S_{\substack{\text{adv.}\\ \text{ret.}}}(x-x';M)\left(\gamma\frac{\partial}{\partial x'} + M\right)\psi(x') \tag{2.2a}$$

$$\overline{\psi}_{\substack{\text{out}\\ \text{in}}}(x) = \overline{\psi}(x) + \int dx'\left(M - \gamma^T\frac{\partial}{\partial x'}\right)\overline{\psi}(x')\, S_{\substack{\text{ret.}\\ \text{adv.}}}(x'-x;M). \tag{2.2b}$$

These equations are meant to hold in the sense of "weak operator convergence" i. e. they hold for matrix elements for the operators between normalizable states. ZIMMERMANN shows that these "in" and "out" operators are just the usual "free field" operators, i. e. that they obey the free field equations of motion and the canonical commutation rules

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[1]) W. ZIMMERMANN, Nuovo Cimento **10**, 597 (1958).

$$(\Box - m^2)\,\varphi_{\substack{\text{out}\\ \text{in}}}(x) = 0 \qquad \left[\varphi_{\substack{\text{out}\\ \text{in}}}(x),\ \varphi_{\substack{\text{out}\\ \text{in}}}(y)\right] = i\,\Delta\,(x-y;\ m) \tag{2.3}$$

$$\left.\begin{aligned} &\left(\gamma\frac{\partial}{\partial x} + M\right)\psi_{\substack{\text{out}\\ \text{in}}}(x) = 0 \\ &\left(-\gamma^T\frac{\partial}{\partial x} + M\right)\overline{\psi}_{\substack{\text{out}\\ \text{in}}}(x) = 0 \end{aligned}\right\} \qquad \left\{\psi_{\substack{\text{out}\\ \text{in}}}(x),\ \overline{\psi}_{\substack{\text{out}\\ \text{in}}}(y)\right\} = iS(x-y;M). \tag{2.4}$$

Furthermore they are asymptotic to the field operators in the sense that

$$\lim_{t=\pm\infty}\ (\Psi, a_\alpha(t)\,\Phi) = (\Psi, a_{\substack{\text{out}\\ \text{in}}}(\alpha)\,\Phi) \tag{2.5}$$

where

$$\begin{aligned} a_\alpha(t) &= -i\int_{x_0=t} d^3x\left(\varphi(x)\frac{\partial}{\partial x_0}f_\alpha^*(x) - \frac{\partial\varphi(x)}{\partial x_0}f_\alpha^*(x)\right) \equiv -i\int_{x_0=t} d^3x\ \varphi(x)\,\overleftrightarrow{\partial_{x_0}}\,f_\alpha^*(x) \\ a_{\substack{\text{out}\\ \text{in}}}(\alpha) &= -i\int d^3x\ \varphi_{\substack{\text{out}\\ \text{in}}}(x)\,\overleftrightarrow{\partial_{x_0}}\,f_\alpha^*(x) \end{aligned} \tag{2.6}$$

with $f_\alpha(x)$ a normalizable positive frequency solution of the Klein-Gordon equation, and with Ψ and Φ representing any two normalizable states in the Hilbert space. This definition of $a_\alpha(t)$ comes from the inversion of the expansion

$$\varphi(x) = \sum_\alpha\left(a_\alpha(x_0)\,f_\alpha(x) + a_\alpha^*(x_0)\,f_\alpha^*(x)\right) \tag{2.7}$$

with the use of the orthonormality conditions

$$-i\int d^3x\ f_\alpha(x)\,\overleftrightarrow{\partial_{x_0}}\,f_\beta^*(x) = \delta_{\alpha\beta}; \qquad \int d^3x\ f_\alpha(x)\,\overleftrightarrow{\partial_{x_0}}\,f_\beta(x) = 0. \tag{2.8}$$

For spinors, the expansion to be inverted is

$$\psi(x) = \sum_\alpha\left(a_\alpha(x_0)\,u_\alpha(x) + b_\alpha^*(x_0)\,v_\alpha(x)\right) \tag{2.9}$$

where $u_\alpha(x)$ and $v_\alpha(x)$ are positive and negative frequency solutions, respectively, of the Dirac equation, with the orthonormality properties

$$\begin{aligned} &\int d^3x\ \overline{u}_\beta(x)\,\gamma_4\,u_\alpha(x) = \delta_{\beta\alpha} \\ &\int d^3x\ \overline{v}_\beta(x)\,\gamma_4\,u_\alpha(x) = 0 \\ &\int d^3x\ \overline{v}_\beta(x)\,\gamma_4\,v_\alpha(x) = \delta_{\beta\alpha}\,. \end{aligned} \tag{2.10}$$

Thus

$$a_\alpha(t) = \int_{x_0=t} d^3x\ \overline{u}_\alpha(x)\,\gamma_4\,\psi(x)$$

has the weak asymptotic limits

$$a_{\substack{\text{out}\\\text{in}}}(\alpha) = \int d^3x\, \bar{u}_\alpha(x)\, \gamma_4\, \psi_{\substack{\text{out}\\\text{in}}}(x) \qquad \text{etc.}$$

The creation and annihilation operators $a^*_{\substack{\text{out}\\\text{in}}}(\alpha)$, $\boldsymbol{a}^*_{\substack{\text{out}\\\text{in}}}(\beta)$, $\boldsymbol{b}^*_{\substack{\text{out}\\\text{in}}}(\gamma), \ldots$; $a_{\substack{\text{out}\\\text{in}}}(\alpha)$, $\boldsymbol{a}_{\substack{\text{out}\\\text{in}}}(\beta)$, $\boldsymbol{b}_{\substack{\text{out}\\\text{in}}}(\gamma), \ldots$ may be used to construct a representation of the Hilbert space²).

The vacuum state is defined by the condition that for all annihilation operators

$$a_{\substack{\text{out}\\\text{in}}}(\alpha)|0\rangle = 0; \quad \boldsymbol{a}_{\substack{\text{out}\\\text{in}}}(\beta)|0\rangle = 0; \quad \boldsymbol{b}_{\substack{\text{out}\\\text{in}}}(\gamma)|0\rangle = 0, \ldots \tag{2.11}$$

The one-particle states are constructed thus

$$\begin{aligned} |1_\alpha\rangle &= a^*_{\text{out}}(\alpha)\,|0\rangle = a^*_{\text{in}}(\alpha)\,|0\rangle \\ |1_\beta\rangle &= \boldsymbol{a}^*_{\text{out}}(\beta)\,|0\rangle = \boldsymbol{a}^*_{\text{in}}(\beta)\,|0\rangle \\ &\ldots\ldots\ldots\ldots \text{etc.} \end{aligned} \tag{2.12}$$

(An arbitrary phase factor is adjusted so that this equality holds.) The remaining states can be constructed in two equivalent ways:

$$\begin{aligned} |\alpha_1\ldots;\beta_1\ldots;\gamma_1\ldots;\text{in}\rangle &= a^*_{\text{in}}(\alpha_1)\ldots\boldsymbol{a}^*_{\text{in}}(\beta_1)\ldots\boldsymbol{b}^*_{\text{in}}(\gamma_1)\ldots|0\rangle \\ |\alpha_1\ldots;\beta_1\ldots;\gamma_1\ldots;\text{out}\rangle &= a^*_{\text{out}}(\alpha_1)\ldots\boldsymbol{a}^*_{\text{out}}(\beta_1)\ldots\boldsymbol{b}^*_{\text{out}}(\gamma_1)\ldots|0\rangle \end{aligned} \tag{2.13}$$

where we have omitted an uninteresting normalization factor. These two complete orthonormal (by construction) bases are connected by the unitary S-matrix. The asymptotic conditions written down above, enable us to express the elements of the S-matrix in terms of certain products of field operators. To study the consequences of local commutativity, it has been found useful to express the S-matrix elements in terms of multiple retarded commutators³). As an illustration, we briefly discuss the pion-nucleon scattering matrix element. We have

$$\begin{aligned} \langle p'q';\text{in}\,|S|\,pq;\text{in}\rangle &= \langle p'q'^{\,\text{out}}\,|pq^{\text{in}}\rangle \\ \langle p'q'^{\,\text{out}}\,|pq^{\text{in}}\rangle &= \langle p'q'^{\,\text{out}}\,|a^*_{\text{in}}(q)|\,p\rangle = \\ &= \operatorname*{Lim}_{t=-\infty} i \int_{y_0=t} d^3y\,\langle p'q'^{\,\text{out}}\,|\varphi(y)|\,p\rangle\, \overleftrightarrow{\partial_{y_0}}\, f_q(y) = \\ &= \operatorname*{Lim}_{t=+\infty} i \int_{y_0=t} d^3y\,\langle p'q'^{\,\text{out}}\,|\varphi(y)|\,p\rangle\, \overleftrightarrow{\partial_{y_0}}\, f_q(y) - \\ &\quad - i\int_{-\infty}^{\infty} dy_0 \frac{\partial}{\partial y_0}\int d^3y\,\langle p'q'^{\,\text{out}}\,|\varphi(y)|\,p\rangle\, \overleftrightarrow{\partial_{y_0}}\, f_q(y). \end{aligned} \tag{2.14}$$

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²) It is of course necessary to use a 'complete set' of such operators. Thus if there are bound states, such as the deuteron, 'in' and 'out' operators for these stable states must also be used in the construction. These bound state 'in' and 'out' operators may occur as asymptotic limits of polynomials of the simple operators $\varphi(x)$, $\psi(x)$. See reference¹).

³) See for example LEHMANN, SYMANZIK and ZIMMERMANN, Nuovo Cimento **6**, 319 (1957).

The first term is clearly $\langle p'q'^{\text{out}}|a^*_{\text{out}}(q)|p\rangle = \langle p'q'^{\text{in}}|pq^{\text{in}}\rangle$. In the second term we carry out the differentiation, and using the fact that $f_q(y)$ is a normalizable solution of the Klein-Gordon equation, we integrate by parts to obtain

$$\langle p'q'^{\text{in}}|S-1|pq^{\text{in}}\rangle = i\int dy\, \langle p'q'^{\text{out}}|j(y)|p\rangle\, f_q(y) \tag{2.15}$$

where we have introduced the definition

$$j(y) = -K^m_y\, \varphi(y) \equiv -(\Box - m^2)\,\varphi(y). \tag{2.16}$$

We may contract with respect to any one of the three remaining particles. If we choose the outgoing meson we write

$$\begin{aligned}\langle p'q'^{\text{in}}|S-1|pq^{\text{in}}\rangle = {} & i\int dy\,\langle p'|a_{\text{out}}(q')\,j(y)|p\rangle\, f_q(y) = \\ & = -i\int dy\,\langle p'|[a_{\text{out}}(q'),\,\varphi(y)]\,p\rangle\, \overleftarrow{K}^m_y\, f_q(y).\end{aligned} \tag{2.17}$$

Then we write

$$\begin{aligned}\langle p'|[a_{\text{out}}(q'),\varphi(y)]|p\rangle &= \lim_{t=+\infty} i\int_{x_0=t} d^3x\, f^*_{q'}(x)\,\overleftrightarrow{\partial}_{x_0}\,\langle p'|[\varphi(x),\,\varphi(y)]|p\rangle = \\ &= \lim_{t=+\infty} i\int_{x_0=t} d^3x\, f^*_{q'}(x)\,\overleftrightarrow{\partial}_{x_0}\,\langle p'|\theta(x-y)\,[\varphi(x),\,\varphi(y)]|p\rangle =\end{aligned}$$

or

$$\begin{aligned}&= \lim_{t=+\infty} i\int_{x_0=t} d^3x\, f^*_{q'}(x)\,\overleftrightarrow{\partial}_{x_0}\,\langle p'|[\varphi(x),\,\varphi(y)]|p\rangle = \\ &= i\int_{x_0=y_0} + i\int dx\,\theta(x-y)\,\frac{\partial}{\partial x_0}\left[f^*_{q'}(x)\,\overleftrightarrow{\partial}_{x_0}\langle p'|[\varphi(x),\varphi(y)]|p\rangle\right].\end{aligned}$$

Carrying out the same manipulations as before we get from the first form

$$\langle p'q'^{\text{in}}|S-1|pq^{\text{in}}\rangle = (-i)^2\iint dx\,dy\, f^*_{q'}(x)\,\overrightarrow{K}^m_y\,\langle p'|\theta(x-y)\,[\varphi(x),\varphi(y)]|p\rangle\,\overleftarrow{K}^m_y\, f_q(y) \tag{2.18}$$

and from the second form

$$\begin{aligned}\langle p'q'^{\text{in}}|S-1|pq^{\text{in}}\rangle = {} & -\iint dx\,dy\, f^*_{q'}(x)\,\langle p'|\theta(x-y)\,[j(x),\,j(y)]|p\rangle\, f_q(y) - \\ & -\iint dx\,dy\;\delta(x_0-y_0)\, f^*_{q'}(x)\,\overleftrightarrow{\partial}_{x_0}\,\langle p'|[\varphi(x),\,j(y)]|p\rangle\, f_q(y).\end{aligned} \tag{2.19}$$

In this form, the second term is an equal-time commutator, which can be evaluated once an interaction and the canonical commutation rules are assumed. For example in the *PS-PS* theory with unrenormalized operators (and mass) we find

$$\begin{aligned}&[\varphi(x),\,j(y)]\,\delta(x_0-y_0) = 0, \\ &\left[\frac{\partial\varphi(x)}{\partial x_0},\,j(y)\right]\delta(x_0-y_0) = 3i\lambda\,\varphi^2(x)\,\delta(x-y),\end{aligned} \tag{2.20}$$

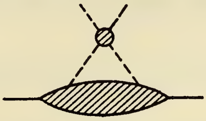
so that the second term is

$$-3i\lambda \int dx\, f_{q'}^{*}(x)\, \langle p' \,|\varphi^2(x)|\, p\rangle\, f_q(x).$$

If we deal with plane wave states, we can carry out the integration, and we get as coefficient of the energy-momentum conservation delta-function

$$-3i\lambda\, \langle p' \,|\varphi^2(o)|\, p\rangle .$$

This term is a function of the momentum transfer $-(p-p')^2$ only, and is associated with graphs of the type



We will see that in a discussion of dispersion relations, these terms need not be evaluated explicitly, as they can be absorbed into the so-called "subtraction terms", which we will discuss in sec. V. For completeness we give the expression for the S-matrix element obtained when the contraction is carried out with respect to the outgoing nucleon. We find that

$$\langle p' q' \;|S-1|\, p q^{\text{in}}\rangle =$$

$$= \int\int dx\, dy\, \overline{u}_{p'}(x)\, \overrightarrow{\left(\gamma \frac{\partial}{\partial x} + M\right)}\, \langle q' \,|\theta(x-y)\, [\psi(x), \varphi(y)]|\, p\rangle\, \overleftarrow{K_y^m}\, f_q(y). \quad (2.21)$$

We can continue with the contractions until we end up with the vacuum expectation value of a generalized retarded commutator. Thus far this has not proved to be a useful form, though it could undoubtedly yield more information than the forms obtained above.

III. The Scattering Amplitude

In practice we want to describe the interacting particles by their initial and final momenta, so that instead of the normalizable wave packets $f_q(x)$ which we considered heretofore, we use plane wave states. One way of getting around the difficulty of the non-normalizability of such states is to quantize the system in a box of volume V with periodic boundary conditions. We may then use

$$f_q(x) = \frac{1}{\sqrt{2\,\omega V}}\, e^{iqx};\; u_p(x) = \sqrt{\frac{M}{EV}}\, u(p)\, e^{ipx};\; v_p(x) = \sqrt{\frac{M}{EV}}\, v(p) e^{-ipx} \quad (3.1)$$

where $qx = \boldsymbol{q}\cdot\boldsymbol{x} - q_0 x_0$; $q_0 = \omega \equiv \sqrt{\boldsymbol{q}^2 + m^2}$; $p_0 = E \equiv \sqrt{\boldsymbol{p}^2 + M^2}$ and $u(p)$ $(v(p))$ are positive (negative) energy solutions of the Dirac equation⁴). The

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[4]) Explicit representations of these functions are

$$u_a^{(r)}(p) = \frac{(M - i\gamma p)_{ar}}{\sqrt{2M\,(M+E)}}\,(r = 1, 2);\quad v_a^{(r-2)}(p) = \frac{(M + i\gamma p)_{ar}}{\sqrt{2M\,(M+E)}}\,(r = 3, 4).$$

momentum variable is now discrete, so that for example

$$\langle p' \mid p \rangle = \delta_{\mathbf{p}', \mathbf{p}}.$$

In the limit $V \to \infty$, the transition from a sum over momenta to the corresponding integral is carried out according to the prescription

$$\sum_{\mathbf{p}} \to \int \frac{d^3p\, V}{(2\pi)^3}.$$

With these wave functions, the meson-nucleon scattering matrix element is, aside from the equal-time commutator terms which we shall omit from now on,

$$-\frac{1}{\sqrt{4\,\omega\,\omega'\,V^2}} \iint dx\, dy\, e^{-iq'x+iqy} \langle p' \,|\, \theta(x-y)\, [j(x),\ j(y)] \,|\, p \rangle. \tag{3.2}$$

Using invariance under translation ($j(x) = e^{-iPx}\, j(0)\, e^{iPx}$ with P the energy-momentum operator) we can write this as follows

$$-\int d\xi\, e^{i\xi(p+q-p'-q')} \frac{1}{\sqrt{4\,\omega\,\omega'\,V^2}} \int dz\, e^{-i\frac{q+q'}{2}z}\, \theta(z) \left\langle p' \left| \left[j\left(\frac{z}{2}\right), j\left(-\frac{z}{2}\right) \right] \right| p \right\rangle. \tag{3.3}$$

As long as V is finite, the time interval should also be kept finite; in fact, to avoid effects caused by reflections at the boundaries, we must take $T << V^{1/3}$. The S-matrix element thus is

$$-\left(\frac{2}{\varDelta} \sin \frac{T\varDelta}{2}\right) (V\,\delta_{\mathbf{p}+\mathbf{q},\,\mathbf{p}'+\mathbf{q}'}) \frac{1}{\sqrt{4\,\omega\,\omega'\,V^2}} \times$$
$$\times \int dz\, e^{-i\frac{q+q'}{2}z}\, \theta(z) \left\langle p' \left| \left[j\left(\frac{z}{2}\right), j\left(-\frac{z}{2}\right) \right] \right| p \right\rangle \tag{3.4}$$

where we have written $\varDelta = p_0 + q_0 - p_0' - q_0'$. As V and $T \to \infty$ we get

$$V\,\delta_{\mathbf{p}+\mathbf{q},\,\mathbf{p}'+\mathbf{q}'} \to (2\pi)^3\, \delta(\mathbf{p}+\mathbf{q}-\mathbf{p}'-\mathbf{q}')$$
$$\frac{2}{\varDelta} \sin \frac{T\varDelta}{2} \to 2\pi\, \delta(\varDelta). \tag{3.5}$$

With our normalization the states $|p\rangle$ and $|p'\rangle$ etc. are not normalized in an invariant way. To see how they transform we notice that had we carried out all contractions until we had been left with a vacuum expectation value of a four-operator product, we would have something of the form

$$\left(\frac{1}{2\omega' V} \frac{M}{E' V} \frac{1}{2\omega V} \frac{M}{E V}\right)^{1/2}$$
$$\overline{u}(p') \int \ldots \int dx_1 \ldots dx_4\, e^{i(px_1+qx_2-p'x_3-q'x_4)}\, \tilde{r}\, (x_1, \ldots x_4)\, u(p)$$

where $\tilde{r}$ is a retarded "amputated" (i. e. multiplied by Dirac and Klein-Gordon operators) product. Thus the general form of the S-matrix element is

$$\langle p'q'^{\text{in}}\,|S-1|\,pq^{\text{in}}\rangle = -\frac{(2\pi)^4\, i\,\delta(p+q-p'-q')}{n_{p'}\, n_{q'}\, n_p\, n_q\, V^2}\, T(p'q';pq) \tag{3.6}$$

where $n_p = \sqrt{\frac{E}{M}}$ for fermions and $n_q = \sqrt{2\omega}$ for bosons. Furthermore the boson-fermion T function will have the form

$$T(p'q';pq) = \bar{u}_\alpha(p')\, I_{\alpha\beta}(p'q';pq)\, u_\beta(p). \tag{3.7}$$

If we assume invariance under space reflections, the most general form for $I_{\alpha\beta}$ is

$$I_{\alpha\beta}(p'q';pq) = -A\,\delta_{\alpha\beta} + \left(i\gamma\,\frac{q+q'}{2}\right)_{\alpha\beta} B \tag{3.8}$$

where A and B are functions of the scalar invariants $p'^2,\ p'\cdot p,\ p'\cdot q, \ldots$ Previously we found that

$$\langle p'q'^{\text{in}}\,|S-1|\,pq^{\text{in}}\rangle = i\int dy\,\langle p'q'^{\text{out}}\,|j(y)|\,p\rangle\,\frac{e^{iqy}}{\sqrt{2\omega V}} =$$

$$= \frac{(2\pi)^4\, i\,\delta(p+q-p'-q')}{n_q\, V^{1/2}}\,\langle p'q'^{\text{out}}\,|j(0)|\,p\rangle. \tag{3.9}$$

Hence

$$T(p'q';pq) = -n_p\, n_{p'}\, n_{q'}\, V^{3/2}\,\langle p'q'^{\text{out}}\,|j|\,p\rangle. \tag{3.10}$$

Generally

$$T(n;\,pq) = -n_p\, N_n\, V^{\frac{n+1}{2}}\,\langle n^{\text{out}}\,|j|\,p\rangle \tag{3.11}$$

where N_n is a product of the normalization factors n_i for the particles in the state $|n\rangle$. To find the relation between the T function and the scattering amplitude, we calculate the cross section for the process $N(p) + \pi(q) \rightarrow$ "n".
From the form

$$\langle n\,|S-1|\,pq\rangle = -\frac{i}{n_p\, n_q N_n\, V^{1+1/2\,n}}\left(\frac{2}{\Delta}\sin\frac{T\Delta}{2}\right)(V\,\delta_{p+q,\,P_n})\; T(n;\,pq)$$

we find that the number of transitions per unit time is

$$\frac{1}{T}\,\frac{1}{n_p^2\, n_q^2 N_n^2\, V^n}\left(\frac{4}{\Delta^2}\sin^2\frac{T\Delta}{2}\right)\delta_{p+q,\,P_n}\,|T(n;\,pq)|^2.$$

With our normalization, the incident flux is $|\boldsymbol{v}_p - \boldsymbol{v}_q|/V$ (where $\boldsymbol{v}_p$ is the velocity of the particle with momentum $\boldsymbol{p}$), so that the cross section is

$$\frac{4\sin^2\dfrac{T\Delta}{2}}{T\Delta^2}\, V\,\delta_{p+q,\,P_n}\,\frac{1}{|\boldsymbol{v}_p - \boldsymbol{v}_q|}\,\frac{|T(n;\,pq)|^2}{n_p^2\, n_q^2\, N_n^2\, V^n}.$$

As $T \to \infty$

$$\frac{4 \sin^2 \dfrac{T\Delta}{2}}{T\Delta^2} \to 2\pi\, \delta(\Delta)$$

(with $\Delta = p_0 + q_0 - P_{n0}$ here), so that the cross section is

$$\frac{(2\pi)^4\, \delta(p + q - P_n)}{n_p^2\, n_q^2\, V^n\, N_n^2} \frac{1}{|\boldsymbol{v}_p - \boldsymbol{v}_q|} |T(n;\, pq)|^2.$$

This is an invariant quantity[5], and in the center of mass system, where

$$|\boldsymbol{v}_p - \boldsymbol{v}_q| = \frac{q}{E_q} + \frac{q}{\omega_q} = \frac{qW}{E\omega} = \frac{2qW}{M} \frac{1}{n_p^2\, n_q^2}$$

with W the total energy,

$$\sigma_n = \frac{(2\pi)^4\, \delta(p + q - P_n)}{N_n^2\, V^n} \frac{M}{2qW} |T(n;\, pq)|^2. \tag{3.12}$$

The total cross section is obtained by summing over all possible final states. We can also calculate the cross section for elastic scattering. We have in the center of mass system

$$d\sigma = \frac{V d^3p'}{(2\pi)^3} \frac{V\, d^3q'}{(2\pi)^3} \frac{(2\pi)^4\, \delta(\boldsymbol{p}' + \boldsymbol{q}')\, \delta(W - E' - \omega')}{(2E'\omega'/M)\, V^2} \frac{M}{2qW} |T_2|^2 =$$

$$= \frac{d^3p'\, d^3q'}{(2\pi)^2} \frac{M^2}{4E'\omega' qW}\, \delta(\boldsymbol{p}' + \boldsymbol{q}')\, \delta(W - E' - \omega')\, |T_2|^2. \tag{3.13}$$

Hence

$$\frac{d\sigma}{d\Omega_{\hat{q}'}} = \int_0^\infty q'^2\, dq' \int d^3p' \frac{M^2}{16\pi^2 E'\omega' qW}\, \delta(\boldsymbol{p}' + \boldsymbol{q}')\, \delta(W - E' - \omega')\, |T_2|^2. \tag{3.14}$$

Since

$$dq' = \frac{d(E' + \omega')}{q'\left(\dfrac{1}{E'} + \dfrac{1}{\omega'}\right)} = \frac{E'\omega'}{q'W}\, d(E' + \omega')$$

we have

$$\frac{d\sigma}{d\Omega_{\hat{q}'}} = \frac{M^2}{16\pi^2 W^2} \int \frac{q'}{q}\, d(E' + \omega')\;\, \delta(E' + \omega' - W)\, |T_2|^2. \tag{3.15}$$

If we write

$$\frac{d\sigma}{d\Omega_{\hat{q}}} = \frac{M^2}{16\pi^2\, W^2} |T_2|^2 = |F|^2 \tag{3.16}$$

we obtain the relation

$$F = -\frac{M}{4\pi W} T_2. \tag{3.17}$$

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[5] C. Møller, Dan. Mat. Fys. Medd. 23 no. 1 (1945).

We shall see that the — sign corresponds to the conventional definition of the scattering amplitude, when we discuss the optical theorem in the next section. Note that for boson-boson scattering we have the form

$$F = -\frac{1}{8\pi W} T \tag{3.18}$$

and for the process $\pi + \pi \to N + \bar{N}$ we have

$$\frac{d\sigma}{d\Omega} = \frac{M^2}{16\pi^2 W^2} \frac{p_f}{p_i} |T|^2 \equiv \frac{p_f}{p_i} |F|^2 \tag{3.19}$$

where p_f is the momentum in the final state (in the center of mass system) and p_i the initial momentum.

IV. The Unitarity of the S Matrix

If we write

$$S = 1 - i\tilde{R} \tag{4.1}$$

then, as seen before, the matrix elements of $\tilde{R}$ have the form

$$\langle n | \tilde{R} | pq \rangle = \frac{(2\pi)^4 \, \delta(p + q - P_n)}{n_p \, n_p \, N_n \, V^{1 + {}^1/{}_2 n}} T(n; pq). \tag{4.2}$$

The unitarity condition is $SS^+ = S^+S = 1$, i. e. $i(\tilde{R}^+ - \tilde{R}) = -\tilde{R}^+\tilde{R}$ so that

$$i \langle p'q' | \tilde{R}^+ - \tilde{R} | pq \rangle = -\sum_n \langle p'q' | \tilde{R}^+ | n \rangle \langle n | \tilde{R} | pq \rangle \tag{4.3}$$

or

$$i \langle pq | \tilde{R} | p'q' \rangle^* - i \langle p'q' | \tilde{R} | pq \rangle = -\sum_n \langle n | \tilde{R} | p'q' \rangle^* \langle n | \tilde{R} | pq \rangle.$$

Hence

$$\frac{(2\pi)^4 \, i \, \delta(p + q - p' - q')}{n_p n_q n_{p'} \, n_{q'} \, V^2} \left(T^* (pq;\, p'q') - T(p'q';\, pq)\right) =$$

$$= -\sum_n \frac{(2\pi)^8 \, \delta(p + q - P_n) \, \delta(p' + q' - P_n)}{n_p n_q n_{p'} \, n_{q'} \, N_n^2 \, V^{2+n}} \, T^* (n; p'q') \, T(n; pq)$$

i. e.

$$i(T^*(pq;\, p'q') - T(p'q';\, pq)) = -(2\pi)^4 \sum_n \frac{\delta(p + q - P_n)}{N_n^2 \; V^n} T^*(n; p'q') \, T(n; pq). \tag{4.4}$$

Note that for an n-particle state, the sum is of the form $\int \frac{V d^3 p_1}{(2\pi)^3} \ldots \int \frac{V d^3 p_n}{(2\pi)^3}$ so that the T's are volume independent.
For forward scattering

$$2 \, \mathrm{Im} \, T(pq;\, pq) = -(2\pi)^4 \sum_n \frac{\delta(p + q - P_n)}{N_n^2 \, V^n} \, |T(n;\, pq)|^2 \tag{4.5}$$

i. e.

$$\operatorname{Im} F(pq;\, pq) = \frac{M}{8\pi W}\, (2\pi)^4 \sum_n \frac{\delta(p + q - P_n)}{N_n^2\, V^n}\, |T\,(n;\, pq)|^2 = \frac{q}{4\pi}\, \sigma_{\text{TOT}} \tag{4.6}$$

which is the well known *optical theorem.*
We shall now show that the unitarity condition contained in the reduction formulae is a generalization of the one obtained above. We found earlier (2.19) that

$$T\,(p'q';\, pq) = -i n_p\, n_{p'}\, V \int dx\, e^{-i\frac{q+q'}{2}x}\, \theta(x) \left\langle p' \left| \left[j_\beta\left(\frac{x}{2}\right), j_\alpha\left(-\frac{x}{2}\right)\right] \right| p \right\rangle. \tag{4.7}$$

From this we can show that

$$T^*(pq;\, p'q') = \\ = i n_{p'}\, n_p\, V \int dx\, e^{-i\frac{q+q'}{2}x}\, \theta(-x) \left\langle p' \left| \left[j_\beta\left(\frac{x}{2}\right), j_\alpha\left(-\frac{x}{2}\right)\right] \right| p \right\rangle \tag{4.8}$$

so that

$$i\,(T^*(pq;\, p'q') - T\,(p'q';\, pq)) = \\ = -\, n_p\, n_{p'}\, V \int dx\, e^{-i\frac{q+q'}{2}x} \left\langle p' \left| \left[j_\beta\left(\frac{x}{2}\right), j_\alpha\left(-\frac{x}{2}\right)\right] \right| p \right\rangle. \tag{4.9}$$

Using invariance under displacements, we can carry out the x-integration after inserting a complete set of states between the operators, and with the identification

$$\langle n\, |j_\alpha|\, p\rangle = -\frac{1}{n_p\, N_n\, V^{\frac{n+1}{2}}}\, T\,(n;\, pq) \tag{4.10}$$

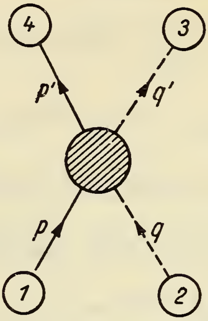
we find that

$$i\,(T^*(pq;\, p'q') - T\,(p'q';\, pq)) = \\ = -\,(2\pi)^4 \sum_n \left\{ \frac{\delta(P_n - p - q)}{N_n^2\, V^n}\, T^*(n;\, p'q')\, T\,(n;\, pq) - \right. \\ \left. - \frac{\delta(P_n - p + q')}{N_n^2\, V^n}\, T^*(n;\, p', -q)\, T\,(n;\, p, -q') \right\}. \tag{4.11}$$

If we compare this with equation (4.4) we see that the first term is identical with that obtained from the condition $SS^+ = 1$. This condition only holds for physical values of the momenta, as it is only for these that the "in" and "out" states are defined and related by the S-matrix. The second term must therefore vanish in the physical region. This is easily seen to be the case: in the center of mass system, the "mass" of the intermediate state is

$$M_n^2 = -P_n^2 = -(p - q')^2 = (E - \omega)^2 - (\mathbf{p} - \mathbf{q}')^2 = (E - \omega)^2 - 2q^2\,(1 + \cos\theta)$$

for real momenta and physical scattering angles this is always less than $(M + m)^2$. In the nonphysical region (imaginary momenta or angles) this relation yields an extension of the unitarity condition. This extension is only meaningful if all the amplitudes can be continued analytically from the physical domain to the one in which the second term in (4.11) does not vanish. Although such analyticity properties have not yet been proved, we shall assume that the extension of the unitarity condition has physical significance. If we represent the scattering process by the graph



then the first term in $i(T^*(pq;\, p'q') - T(p'q';\, pq))$ contributes when "1" and "2" represent the physical incoming particles, and the second term contributes when "1" and "3" are the incoming particles with physical momenta. We might expect another contribution when "1" and "4" represent the incoming particles and this indeed turns out to be true. To show this, we use the reduction formula (2.21) in which the contraction is made with respect to the outgoing nucleon, and a similar one in which the contraction is made with respect to the ingoing nucleon (and outgoing meson). The latter yields

$$T(p'q';\, pq) =$$

$$= i n_{p'} n_q V \int dx\, e^{-i\frac{p+q'}{2}x}\, \theta(x) \left\langle p' \left| \left[j_\beta\left(\frac{x}{2}\right), \bar{\mathfrak{D}}\left(-\frac{x}{2}\right)\right] \right| q \right\rangle u(p)$$

and the former yields

$$T^*(pq;\, p'q') =$$

$$= -i n_{p'} n_q V \int dx\, e^{-i\frac{p+q'}{2}x}\, \theta(-x) \left\langle p' \left| \left[j_\beta\left(\frac{x}{2}\right), \bar{\mathfrak{D}}\left(-\frac{x}{2}\right)\right] \right| q \right\rangle u(p)$$

with

$$\mathfrak{D}(x) = -\left(\gamma \frac{\partial}{\partial x} + M\right) \psi(x).$$

We thus can show that

$$i(T^*(pq;\, p'q') - T(p'q';\, pq)) =$$

$$= n_{p'} n_q V (2\pi)^4 \sum_n \{ \delta(P_n - p - q) \langle p' | j_\beta | n \rangle \langle n | \bar{\mathfrak{D}} | q \rangle -$$

$$- \delta(P_n + p - p') \langle p' | \bar{\mathfrak{D}} | n \rangle \langle n | j_\beta | q \rangle \} u(p)$$

and using the easily derived relations

$$\langle n\,|\bar{\mathfrak{D}}|\,q\rangle\, u(p) = \frac{T(n;pq)}{n_q\, N_n\, V^{\frac{n+1}{2}}}\,;\ \langle p'\,|\bar{\mathfrak{D}}|\,n\rangle\, u(p) = \frac{T^*(n;p',-p^c)}{n_{p'} N_n\, V^{\frac{n+1}{2}}} \tag{4.12}$$

where the superscript ^{c} denotes an antinucleon state of given energy-momentum[5a], we get

$$\begin{aligned} &i\,(T^*(pq;p'q') - T(p'q';pq)) = \\ &= -(2\pi)^4 \sum_n \left\{ \frac{\delta(P_n - p - q)}{N_n^2\, V^n}\, T^*(n;p'q')\, T(n;pq) - \right. \\ &\left. - \frac{\delta(P_n + p - p')}{N_n^2\, V^n}\, T^*(n;\, p',\, -p^c)\, T(n;\, q,\, -q') \right\}. \end{aligned} \tag{4.13}$$

Again the second term yields an extension of the unitarity condition into a non-physical region.

We now examine the domains in which $i(T^*(pq;p'q') - T(p'q';pq))$ does not vanish. This will happen whenever one of the following does not vanish:

$$\sum_n \frac{\delta(P_n - p - q)}{N_n^2\, V^n}\, T^*(n;\, p'q')\, T(n;\, pq) \tag{4.14a}$$

$$\sum_n \frac{\delta(P_n - p + q')}{N_n^2\, V^n}\, T^*(n;\, p',\, -q)\, T(n;\, p,\, -q') \tag{4.14b}$$

$$\sum_n \frac{\delta(P_n + p - p')}{N_n^2\, V^n}\, T^*(n;\, p',\, -p^c)\, T(n;\, q,\, -q'). \tag{4.14c}$$

We shall express the boundaries in terms of the invariants

$$\begin{aligned} s &= -(p+q)^2 = -(p'+q')^2 \\ t &= -(p-p')^2 = -(q-q')^2 \\ u &= -(p-q')^2 = -(p'-q)^2 \end{aligned} \tag{4.15}$$

which are connected by the relation

$$s + t + u = 2(M^2 + m^2). \tag{4.16}$$

In the center of mass system for process I

$$N_p + \pi_q \to N_{p'} + \pi_{q'}$$

the s and t variables have the following significance:

$$\begin{aligned} s &= (E+\omega)^2 = W^2 \\ t &= -(q-q')^2 = -2q^2(1-\cos\theta_{\mathrm{I}}). \end{aligned} \tag{4.17}$$

[5a] The second of the relations (4.12) should contain a kinematical factor $\sum_\beta \bar{v}_\beta^{(r)}(-p)\, u_\beta^{(\varrho)}(p)$ which we have omitted for the sake of brevity.

In the physical region for this process

$$s \geq (M+m)^2 \tag{4.18}$$

$$|\cos\theta_{\mathrm{I}}| \leq 1$$

using

$$\vec{q}^{\,2} = \frac{[s-(M+m)^2]\,[s-(M-m)^2]}{4s}$$

we get

$$0 \geq t \geq -s + 2(M^2+m^2) - \frac{(M^2-m^2)^2}{s}$$

i. e.

$$t \leq 0; \; u \leq \frac{(M^2-m^2)^2}{s} \leq (M-m)^2. \tag{4.19}$$

Now the masses of the accessible intermediate states in (4.14b, c) are

$$(M_n^2)_{\mathrm{b}} = -(P_n^2)_{\mathrm{b}} = -(p-q')^2 = u$$

$$(M_n^2)_{\mathrm{c}} = -(P_n^2)_{\mathrm{c}} = -(p-p')^2 = t$$

so that the condition (4.19) shows that *no* states $|n\rangle$ exist, for which (4.14b, c) are nonvanishing, since we must have

$$(M_n^2)_{\mathrm{b}} \geq (M+m)^2$$

$$(M_n^2)_{\mathrm{c}} \geq 4M^2$$

in the physical region. This is in agreement with what we found before.

The second term differs from the first only in the interchange $q \leftrightarrow -q'$, i. e. (4.14b) looks like the imaginary part of T for the process

$$\mathrm{N}_p + \pi_{-q'} \rightarrow \mathrm{N}_{p'} + \pi_{-q}.$$

In the center of mass system for this reaction the scattering angle is again determined by t, but the total energy is simply related to u instead of s. Thus the conditions are

$$u \geq (M+m)^2$$

$$0 \geq t \geq -u + 2(M^2+m^2) - \frac{(M^2-m^2)^2}{u}$$

which in terms of s and t are

$$\text{for } s \leq 0 \quad t \leq 0$$

$$\text{for } s > 0 \quad 0 \geq t \geq 2(M^2+m^2) - s - \frac{(M^2-m^2)^2}{s} \tag{4.20}$$

and

$$s + t \leq (M-m)^2. \tag{4.21}$$

The third term is the imaginary term of T for the process

$$\pi_q + \pi_{-q'} \to \mathrm{N}_{p'} + \overline{\mathrm{N}}_{-p}.$$

We introduce a center of mass system for this process

$$\begin{aligned} q &= (\boldsymbol{q}, \omega) \qquad & p' &= (-\boldsymbol{p}, E) \\ -q' &= (-\boldsymbol{q}, \omega) \qquad & -p &= (\boldsymbol{p}, E) \end{aligned} \tag{4.22}$$
$$\omega = \sqrt{q^2 + m^2} = E = \sqrt{p^2 + M^2}.$$

Hence

$$\begin{aligned} s &= -(p+q)^2 = -(\boldsymbol{p}-\boldsymbol{q})^2 = -p^2 - q^2 + 2pq\cos\theta \\ u &= -(p'-q)^2 = -(\boldsymbol{p}+\boldsymbol{q})^2 = -p^2 - q^2 - 2pq\cos\theta \\ t &= -(q-q')^2 = (2\omega)^2. \end{aligned} \tag{4.23}$$

The first possible intermediate state is a two-pion state, for which $t \geq 4m^2$ but no physical process can occur until the intermediate state can go into a physical nucleon-antinucleon state ($\boldsymbol{p}$ real), i. e. we require that

$$t \geq 4M^2. \tag{4.24}$$

The condition that the angles be real is that

$$\begin{aligned} & s \leq 0 \\ & t \geq 2(M^2 + m^2) - s - \frac{(M^2 - m^2)^2}{s}. \end{aligned} \tag{4.25}$$

The boundaries determined above are shown in fig. 1.
There are other domains in the $s - t$ plane in which $\operatorname{Im} T$ does not vanish. These are domains in which the initial and/or final states are nonphysical, but which are allowed to proceed (with energy-momentum conservation) to physical intermediate states⁶).
These regions ar

$$\begin{aligned} s &\geq (M+m)^2 \\ u &\geq (M+m)^2 \\ t &\geq 4m^2 \end{aligned}$$

We must also consider possible one-particle states which can connect the initial to the final states. The momenta must lie in the nonphysical region as otherwise the intermediate particle could not be stable — it would decay into the initial or final state. In the case of the meson-nucleon system there is no stable particle having the same quantum number as a pair of pions, so that there is no inter-

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⁶) Again we must assume that the matrix elements connecting the physical intermediate states to unphysical initial (or final) states can be obtained by analytic continuation from the domain in which the latter are physical.

mediate one-particle state for (4.14c) i. e. process III. In (4.14a, b), i. e. for processes I and II, the one-nucleon state is a possible intermediate state. Thus in (4.14a) we will have a contribution at $s = M^2$ and in (4.14b) a contribution at

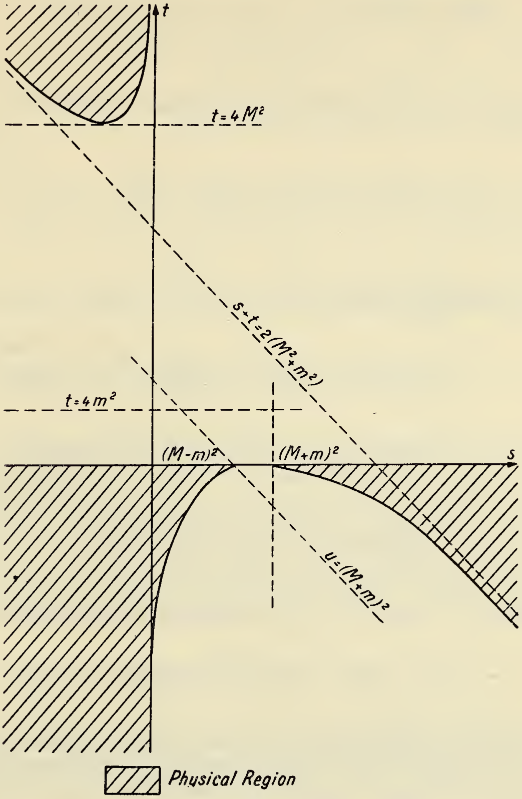

Fig. 1.

$u = M^2$. For later applications we calculate $i\,(T^*(pq;\, p'q') - T(p'q';\, pq))$ at these points. For process I we have

$$i\,(T^*(pq;\, p'q') - T(p'q';\, pq)) =$$

$$= -(2\pi)^4 \int \frac{V\, d^3k}{(2\pi)^3}\, \delta(\mathbf{p}+\mathbf{q}-\mathbf{k})\, \delta\left(E+\omega-\sqrt{\mathbf{k}^2+M^2}\right) \frac{M}{VE_K}\, T^*(k;p'q')\, T(k,pq) =$$

$$= -2\pi \frac{M}{E_{\mathbf{p}+\mathbf{q}}}\, \delta\left(E+\omega-\sqrt{(\mathbf{p}+\mathbf{q})^2+M^2}\right) T^*(p+q;\, p'q')\ T(p+q;\, pq) =$$

$$= -4\pi\, M\, \delta(s-M^2)\, (T^*T)_{\mathrm{I}}. \tag{4.26}$$

To evaluate $(T^*T)_{\mathrm{I}}$ we recall that

$$\begin{aligned} T(p+q;\, pq) &= -n_p n_{p+q} \langle p+q\,|j_\alpha|\,p\rangle \\ T^*(p+q;\, p'q') &= -n_{p'}\, n_{p+q} \langle p+q\,|j_\beta|\,p'\rangle^* \\ &= -n_{p'}\, n_{p+q} \langle p'\,|j_\beta|\,p+q\rangle . \end{aligned}$$

For a pseudoscalar symmetrically coupled meson, we write on grounds of invariance

$$\langle p+q\,|j_\alpha|\,p\rangle = \frac{1}{n_p n_{p+q} V}\, \overline{u}(p+q)\, \gamma_5\, \tau_\alpha\, u(p)\, F(-p^2,\ -(p+q)^2, -q^2) \tag{4.27}$$

so that

$$\begin{aligned} (T^*T)_{\mathrm{I}} &= \sum_{s=1}^{2} \overline{u}(p')\, \gamma_5 \tau_\beta \overset{(s)}{u}(p+q)\, \overset{(s)}{\overline{u}}(p+q)\, \gamma_5 \tau_\alpha\, u(p)\, |F|^2 = \\ &= |F|^2\, \overline{u}(p')\, \gamma_5 \tau_\beta\, \frac{M - i\gamma(p+q)}{2M}\, \gamma_5 \tau_\alpha\, u(p) = \\ &= |F|^2\, \overline{u}(p')\, \tau_\beta\, \tau_\alpha\, \frac{M + i\gamma p + i\gamma q}{2M}\, u(p) = \\ &= \frac{|F|^2}{2M}\, \overline{u}(p')\, \tau_\beta\, \tau_\alpha \left(i\gamma\, \frac{q+q'}{2} \right) u(p) . \end{aligned} \tag{4.28}$$

?

The function $F(-p^2, -(p+q)^2, -q^2)$ has all of its variables on the mass shell

$$-p^2 = M^2;\ -(p+q)^2 = M^2;\ -q^2 = m^2.$$

From its definition in (4.27) we see that, in analogy with the charge in electrodynamics, $F(M^2, M^2, m^2)$ is just the renormalized meson-nucleon coupling constant in a *PS–PS* Yukawa theory. To stress this, we shall write "g" instead of $F(M^2, M^2, m^2)$.

$(T^*T)_{\mathrm{II}}$ has an overall minus sign in front of it, and otherwise differs from $(T^*T)_{\mathrm{I}}$ only in the change $(q, \alpha) \leftrightarrow (-q', \beta)$. Recalling that we wrote (3.8)

$$T(p'q';\ pq) = \overline{u}(p') \left[-A + i\gamma\, \frac{q+q'}{2}\, B \right] u(p)$$

so that

$$T^*(pq;\, p'q') = \overline{u}(p') \left[-A^* + i\gamma\, \frac{q+q'}{2}\, B^* \right] u(p)$$

we find that

$$i\,(T^*(pq;\, p'q') - T(p'q';\, pq)) = \overline{u}(p') \left[-2\,\mathrm{Im}\, A + i\gamma\, \frac{q+q'}{2} \cdot 2\,\mathrm{Im}\, B \right] u(p)$$

i. e.

$$\bar{u}(p')\left[-2\,\mathrm{Im}\,A + i\gamma\,\frac{q+q'}{2}\,2\,\mathrm{Im}\,B\right]u(p) =$$

$$= -4\pi M\,\delta(M^2-s)\,\frac{g^2}{2M}\,\bar{u}(p')\,\tau_\beta\tau_\alpha\,\frac{i\gamma(q+q')}{2}\,u(p) -$$

$$-\,4\pi M\,\delta(M^2-u)\,\frac{g^2}{2M}\,\bar{u}(p')\,\tau_\alpha\tau_\beta\,\frac{i\gamma(q+q')}{2}\,u(p). \qquad (4.29)$$

In general we write

$$B_{\beta\alpha} = B^{(+)}\,\delta_{\beta\alpha} + \frac{1}{2}[\tau_\beta, \tau_\alpha]B^{(-)} = \frac{1}{2}\,\tau_\beta\tau_\alpha\,(B^{(+)}+B^{(-)}) + \frac{1}{2}\,\tau_\alpha\tau_\beta\,(B^{(+)}-B^{(-)}) \qquad (4.30)$$

and similarly for $A_{\beta\alpha}$. Thus as far as the one-particle contributions are concerned, we have

$$\begin{aligned} \mathrm{Im}\,A^{(\pm)} &= 0 \\ \mathrm{Im}\,B^{(\pm)} &= -\pi g^2\,[\delta(M^2-s) \pm \delta(M^2-u)]. \end{aligned} \qquad (4.31)$$

The one-particle singularities will turn out to play a special role in the scattering amplitude. They exhibit explicitly the symmetries of the interaction and they give through the magnitude of "g" a measure of the coupling strength. It is important to note that they were derived without the assumption of a primitive Yukawa coupling: unless forbidden by selection rules, one-particle terms will always appear in the unitarity condition, even though there is no "primitive" graph describing them in a perturbation treatment of a model Hamiltonian.

V. Analyticity Properties of Scattering Amplitudes

The unitarity condition gives rise to a set of nonlinear relations connecting different S-matrix elements, which except for their dependence on the symmetries of the interaction, are model independent. Another important part of the structure of the S-matrix, also model independent, follows from the condition of *local commutativity*, e. g.

$$[j_\alpha(x), j_\beta(y)] = 0 \qquad (x-y)^2 > 0. \qquad (5.1)$$

The fact that the scattering amplitude is related to the FOURIER transform of a function vanishing outside the light cone has some very important implications[7]) which we proceed to study. Consider, for example the integral which appears in the expression for the meson nucleon scattering amplitude

$$\int dx\, e^{-i\frac{q+q'}{2}x}\,\theta(x)\left\langle p'\left|\left[j_\beta\!\left(\frac{x}{2}\right), j_\alpha\!\left(-\frac{x}{2}\right)\right]\right|p\right\rangle.$$

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[7]) GELL-MANN, GOLDBERGER and THIRRING, Phys. Rev. **95**, 1612 (1954). M. GOLDBERGER, Phys. Rev. **99**, 979 (1955).

If we evaluate this in a particularly convenient coordinate system (Breit system)

$$p = \left(\frac{1}{2}\Delta,\ E \equiv \sqrt{M^2 + \frac{1}{4}\Delta^2}\right);\ p' = \left(-\frac{1}{2}\Delta, E\right)$$

$$q = \left(k - \frac{1}{2}\Delta,\ \omega = \sqrt{m^2 + \frac{1}{4}\Delta^2 + k^2}\right);\ q' = \left(\frac{1}{2}\Delta, \omega\right) \tag{5.2}$$

$$\boldsymbol{k}\cdot\Delta = 0$$

we obtain

$$\int d^3x \int_0^\infty dx_0\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega x_0} \left\langle -\frac{\Delta}{2}\middle|\left[j_\beta\left(\frac{x}{2}\right), j_\alpha\left(-\frac{x}{2}\right)\right]\middle|\frac{\Delta}{2}\right\rangle$$

which, because of local commutativity, can also be written thus

$$\int_0^\infty dx_0 \int_0^{x_0} x^2\, dx \int d\Omega_{\hat{x}}\, e^{i\omega x_0}\, e^{-i\hat{k}\cdot x\sqrt{\omega^2-m^2-\frac{1}{4}\Delta^2}}$$

$$\left\langle -\frac{\Delta}{2}\middle|\left[j_\beta\left(\frac{x}{2}\right), j_\alpha\left(-\frac{x}{2}\right)\right]\middle|\frac{\Delta}{2}\right\rangle. \tag{5.3}$$

With this choice of coordinate system, the entire dependence on the variable lies in the exponent, and we may study the properties of the integral when Δ is held fixed and ω is allowed to become complex. Before doing this, it is necessary to separate kinematical factors which come from the spin of the nucleon.

The most general form for the matrix element in the integrand is

$$\left\langle -\frac{\Delta}{2}\middle|\left[j_\beta\left(\frac{x}{2}\right), j_\alpha\left(-\frac{x}{2}\right)\right]\middle|\frac{\Delta}{2}\right\rangle = \frac{M}{E}\,\bar{u}\left(-\frac{\Delta}{2}\right)\mathfrak{M}_{\beta\alpha}\, u\left(\frac{\Delta}{2}\right) \tag{5.4}$$

with

$$\mathfrak{M}_{\beta\alpha} = -\tilde{A}_{\beta\alpha}(x^2, x_0, \Delta^2, \boldsymbol{x}\cdot\Delta) + i\gamma x\, \tilde{B}_{\beta\alpha}(x^2, x_0, \Delta^2, \boldsymbol{x}\cdot\Delta). \tag{5.5}$$

Since $\bar{u}\left(-\frac{\Delta}{2}\right) u\left(\frac{\Delta}{2}\right) = \frac{E}{M}$ and $\bar{u}\left(-\frac{\Delta}{2}\right) i\gamma x\, u\left(\frac{\Delta}{2}\right) = -x_0 - \frac{i}{2M}(\sigma\times\Delta)\cdot\boldsymbol{x}$

we can write

$$\int_0^\infty dx_0 \int_0^{x_0} x^2\, dx \int d\Omega_{\hat{x}}\, e^{i\omega x_0 - i\boldsymbol{k}\cdot\boldsymbol{x}} \left(-\tilde{A} + \frac{M}{E}\left(-x_0 - \frac{i}{2M}\sigma\times\Delta\cdot\boldsymbol{x}\right)\tilde{B}\right) =$$

$$= -\int\ldots\tilde{A} - \frac{M}{E}\left(\frac{1}{i}\frac{\partial}{\partial\omega} + \frac{i}{2M}\sigma\cdot\Delta\times i\nabla_k\right)\int\ldots\tilde{B}$$

and using $\frac{\partial}{\partial\omega} = 2\omega\frac{\partial}{\partial\omega^2}$ and $\nabla_k = 2\boldsymbol{k}\frac{\partial}{\partial\omega^2}$ we get

$$-\int\ldots\tilde{A} - \frac{M}{E}\left(\omega - \frac{i}{2M}\sigma\cdot\Delta\times\boldsymbol{k}\right)\frac{2}{i}\frac{\partial}{\partial\omega^2}\int\ldots\tilde{\boldsymbol{B}}.$$

Now in the Breit system

$$T = \bar{u}\left(-\frac{\Delta}{2}\right)\left[-A + i\gamma\frac{q+q'}{2}B\right]u\left(\frac{\Delta}{2}\right) =$$
$$= -\frac{E}{M}A + \left(-\omega - \frac{i}{2M}\,\sigma\cdot\boldsymbol{k}\times\Delta\right)B \tag{5.6}$$

so that we can identify

$$A = \frac{M}{E}\int\ldots\tilde{A}$$
$$B = -2i\,\frac{M}{E}\,\frac{\partial}{\partial\omega^2}\int\ldots\tilde{B} \tag{5.7}$$

and treat the functions A and B separately. Each is represented by an integral of the following form

$$\int_0^\infty dx_0 \int_0^{x_0} x^2\,dx \int d\Omega_{\hat{x}}\; e^{i\omega x_0}\; e^{-i\hat{n}\cdot\boldsymbol{x}\sqrt{\omega^2 - m^2 - \frac{1}{4}\Delta^2}}\; F(x^2, x_0, \Delta^2, \boldsymbol{x}\cdot\Delta). \tag{5.8}$$

Consider this integral in a frame in which $\hat{n} = \boldsymbol{k}/|\boldsymbol{k}|$ is chosen to lie in the z-direction and Δ in the x-direction. It is clear that the azimuthal integration will leave only those parts of F which are even in $\Delta\cdot\boldsymbol{x}$. The remaining angular integration will only give a contribution from the terms even in $\hat{n}\cdot\boldsymbol{x}$, i. e. only $\cos\hat{n}\cdot\boldsymbol{x}\sqrt{\omega^2 - m^2 - {}^1/_4\,\Delta^2}$ contributes. Thus we see that the square root in the exponent does not give rise to a branch point. If we let ω become complex, the integral will not diverge provided

$$\operatorname{Im}\omega > 0 \tag{5.9}$$

and

$$|\operatorname{Im}\omega| > \left|\operatorname{Im}\sqrt{\omega^2 - m^2 - \frac{1}{4}\Delta^2}\right|$$

i. e.

$$\frac{1}{4}\Delta^2 < -m^2. \tag{5.10}$$

This unfortunately never happens. If, however, we consider the meson momenta to be "unphysical" in the sense that they are real but such that $q^2 = q'^2 = -\zeta$ with ζ sufficiently negative, then for such values of the momenta we can establish analyticity of the scattering amplitude in the upper half ω plane[8]). We shall return to the difficult point of justifying this result for "physical" mesons later. What we have established so far is that for $\zeta < -{}^1/_4\,\Delta^2$ the quantities $A_{\beta\alpha}(\omega, \Delta, \zeta)$ and $B_{\beta\alpha}(\omega, \Delta, \zeta)$ are analytic functions of ω for Δ fixed and $\operatorname{Im}\omega > 0$.

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[8]) This trick is due to N. Bogoliubov. See Bogoliubov, Medvedev and Polivanov, lecture notes. Translated at the Institute for Advanced Study, Princeton (1957); in German: Fortschr. Phys. 6, 169—245 (1958).

In exactly the same way we can show that $A^*_{\beta\alpha}(\omega, \Delta, \zeta)$ and $B^*_{\beta\alpha}(\omega, \Delta, \zeta)$ which are contained in

$$\int dx\, e^{+i\frac{q+q'}{2}x}\,\theta(-x)\left\langle p'\left|\left[j_\beta\left(\frac{x}{2}\right), j_\alpha\left(-\frac{x}{2}\right)\right]\right|p\right\rangle$$

are analytic functions of ω for Δ fixed and $\operatorname{Im}\omega < 0$. Thus the scalar functions making up

$$T_{\beta\alpha}(\omega, \Delta, \zeta) = -i\left(-\frac{E}{M}A_{\beta\alpha} + \left(-\omega + \frac{i}{2M}\sigma\cdot\boldsymbol{k}\times\Delta\right)B_{\beta\alpha}\right) \tag{5.11}$$

and

$$T^*_{\beta\alpha}(\omega, \Delta, \zeta) = i\left(-\frac{E}{M}A^*_{\beta\alpha} + \left(-\omega + \frac{i}{2M}\sigma\cdot\boldsymbol{k}\times\Delta\right)B^*_{\beta\alpha}\right) \tag{5.12}$$

are boundary values of analytic functions. We now make use of the mass spectrum condition exhibited in the unitarity condition, namely that $T^*_{\beta\alpha}(\omega, \Delta, \zeta) - T_{\beta\alpha}(\omega, \Delta, \zeta)$ vanishes along part of the real ω axis. In fact, except for one-particle contributions in $B_{\beta\alpha}$, the functions A, B and A^*, B^* respectively, are equal for

$$(M+m)^2 > -(p+q)^2 = -\boldsymbol{k}^2 + (E+\omega)^2 = E^2 + 2E\omega + \frac{1}{4}\Delta^2 + \zeta$$

i. e.

$$\omega < \frac{2Mm + m^2 - \frac{1}{2}\Delta^2 - \zeta}{2E} \tag{5.13}$$

and

$$(M+m)^2 > -(p-q')^2 = -\boldsymbol{k}^2 + (E-\omega)^2 = E^2 - 2E\omega + \frac{1}{4}\Delta^2 + \zeta$$

i. e.

$$\omega > \frac{-2Mm - m^2 + \frac{1}{2}\Delta^2 + \zeta}{2E}. \tag{5.14}$$

Thus A and A^* (and B and B^*) are different boundary values of a single function analytic in the entire ω plane, except for cuts on the real axis extending from $\pm(2Mm + m^2 - {}^1/_2\Delta^2 - \zeta)/2E$ to $\pm\infty$ (again excluding the one-particle terms in B). We can thus write

$$\begin{aligned} A^{(\pm)}(\omega, \Delta, \zeta) &= \frac{1}{2\pi i}\int_C \frac{d\omega'}{\omega' - \omega} A^{(\pm)}(\omega', \Delta, \zeta) \\ B^{(\pm)}(\omega, \Delta, \zeta) &= \frac{1}{2\pi i}\left(\int_C + \int_{C_1}\right)\frac{d\omega'}{\omega' - \omega} B^{(\pm)}(\omega', \Delta, \zeta) \end{aligned} \tag{5.15}$$

with the contours C and C_1 shown on fig. 2. We have written down no contribution to the CAUCHY integral from the contour at infinity. This form therefore corresponds to the assumption that the functions A and B vanish in the limit $|\omega| \to \infty$. The behavior of these functions at infinity is a priori unknown. It depends on the degree of the singularity of the commutator of the field operators on the light cone. For a given model, this can in principle be calculated, though so far this has only been possible in perturbation theory. One assumes, because of the locality of the theory, that the singularity in x-space is no worse than a delta-function or finite derivatives thereof, i. e. A and B should increase no faster than some finite power of ω as $|\omega| \to \infty$. As mentioned in sec. II, the equal-time commutator terms in the reduction formulae provide contributions of a type to make the above integrals diverge. If these are the only such terms, then the renormalizability of the theory in a perturbation sense would provide very definite limitations on the behavior of the amplitude at infinity. We do not know, however, that this is the case, and we must implicitly, at least, recognize that the above relations may have to be written for quantities like

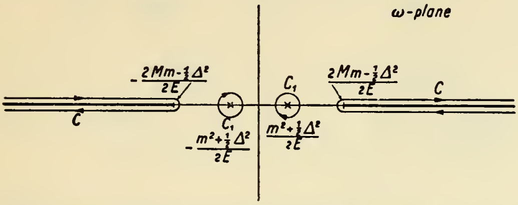
ω-plane
$-\frac{2Mm-\frac{1}{2}\Delta^2}{2E}$
$\frac{2Mm-\frac{1}{2}\Delta^2}{2E}$
C
C
C_1
C_1
$-\frac{m^2+\frac{1}{2}\Delta^2}{2E}$
$\frac{m^2+\frac{1}{2}\Delta^2}{2E}$

Fig. 2. Location of cuts and poles in the complex ω plane.

$$A^{(\pm)}(\omega, \Delta) - \sum_{n=0}^{N} C_n^{(\pm)}(\Delta)\omega^n. \tag{5.16}$$

In the actual writing of equations we shall omit the polynomial, but until we return to a discussion of it in sec. VII, we must always admit its possible existence.

With $\omega_0 = \dfrac{2Mm - \frac{1}{2}\Delta^2 + m^2 - \zeta}{2E}$ we have

$$A^{(\pm)}(\omega, \Delta) = \frac{1}{2\pi i}\left(\int_{-\infty+i\varepsilon}^{-\omega_0} + \int_{-\omega_0}^{-\infty-i\varepsilon} + \int_{\infty-i\varepsilon}^{\omega_0} + \int_{\omega_0}^{\infty+i\varepsilon}\right)\frac{d\omega'}{\omega'-\omega} A^{(\pm)}(\omega', \Delta)$$

and since

$$\begin{aligned} \lim_{\varepsilon\to 0+} A^{(\pm)}(\omega' + i\varepsilon, \Delta) &= A^{(\pm)}(\omega', \Delta) \\ \lim_{\varepsilon\to 0+} A^{(\pm)}(\omega' - i\varepsilon, \Delta) &= (A^{(\pm)}(\omega', \Delta))^* \end{aligned} \tag{5.17}$$

we get

$$A^{(\pm)}(\omega, \Delta, \zeta) = \frac{1}{\pi}\int_{\omega_0}^{\infty}\frac{d\omega'}{\omega'-\omega}\,\mathrm{Im}\,A^{(\pm)}(\omega', \Delta, \zeta) + \frac{1}{\pi}\int_{-\infty}^{-\omega_0}\frac{d\omega'}{\omega'-\omega}\,\mathrm{Im}\,A^{(\pm)}(\omega', \Delta, \zeta). \tag{5.18}$$

The $B^{(\pm)}$ equations have extra terms. Since

$$\operatorname{Im} B^{(\pm)}(\omega, \Delta, \zeta) = -\pi\, |g(\zeta)|^2\, (\delta(M^2 - s) \pm \delta(M^2 - u)) =$$

$$= -\frac{\pi\, |g(\zeta)|^2}{2E}\left(\delta\left(\omega + \frac{\frac{1}{2}\Delta^2 + \zeta}{2E}\right) \pm \delta\left(\omega - \frac{\frac{1}{2}\Delta^2 + \zeta}{2E}\right)\right)$$

we get

$$B^{(\pm)}(\omega, \Delta, \zeta) = -\frac{|g(\zeta)|^2}{2E\omega + \frac{1}{2}\Delta^2 + \zeta} \pm \frac{|g(\zeta)|^2}{\frac{1}{2}\Delta^2 + \zeta - 2E\omega} +$$

$$+ \frac{1}{\pi}\int_{\omega_0}^{\infty} \frac{d\omega'}{\omega' - \omega} \operatorname{Im} B^{(\pm)}(\omega', \Delta, \zeta) + \frac{1}{\pi}\int_{-\infty}^{-\omega_0} \frac{d\omega'}{\omega' - \omega} \operatorname{Im} B^{(\pm)}(\omega', \Delta, \zeta). \tag{5.19}$$

Note that we wrote $g(\zeta)$: this is because the coefficient of the one-particle terms was $F(M^2, M^2, -q^2)$, and so far we have $-q^2 = \zeta$. To prove the dispersion relation for "physical" mesons, we define $B^{(\pm)}(\omega, \Delta, \zeta)$ as a function of ζ by the r. h. s. of eq. (5.19). If we could prove that $\operatorname{Im} B^{(\pm)}(\omega', \Delta, \zeta)$ (and $g(\zeta)$) can be analytically continued from $\zeta < -{}^1/_4\Delta^2$ to $\zeta = m^2$, then $B^{(\pm)}(\omega, \Delta)$ could be defined as the analytic continuation of the r. h. s. of (5.19) to $\zeta = m^2$. To establish the necessary analyticity properties, one may use the theory of functions of many complex variables[9]), but it is simpler *for this problem* to use a representation for the matrix elements of a retarded commutator due to Jost and Lehmann, generalized by Dyson[10]). With the use of this representation Lehmann has shown that the necessary analytic continuation can be made provided that

$$\Delta^2 \leq \frac{8}{3}\, m^2\, \frac{2M + m}{2M - m}. \tag{5.20}$$

This condition could probably be improved by a study of the four-point Green's function, allowing all invariants to become complex. For the proof of the forward scattering dispersion relations, the only ones to have been studied in detail, this formidable program is unnecessary, and for the proof of the analyticity properties which will be assumed later, it is probably insufficient.

VI. Forward Scattering Dispersion Relations

The relations derived in the previous section suffer from a major defect: in the subtraction polynomial arbitrary functions of Δ, the momentum transfer, appear, and this makes it impossible to work out an approximation scheme based on these dispersion relations[11]). For $\Delta = 0$, i. e. for forward scattering, we can make use of the optical theorem (4.6) to obtain relations between the forward scattering amplitude and certain integrals over the total cross section. We will discuss these in some detail. As $\Delta \to 0$, the Breit system reduces to the laboratory system. There

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[9]) See ref. [8]), also Bremermann, Oehme and Taylor, Phys. Rev. 109, 2178 (1958). For a "physicists' summary" see parts of Källen and Wightman, Mat. Fys. Dan. Vid. Selsk. 1, no. 6 (1958).

[10]) A good summary of the proof with the use of the representation can be found in H. Lehmann, Suppl. Nuovo Cim. 14, 153 (1959).

$$p = p' = (0, M); \quad q = q' = (\mathbf{q}, \omega)$$

$$\sigma_{\mathrm{TOT}} = (2\pi)^4 \frac{\omega}{q} \sum_n \frac{\delta(P_n - p - q)}{N_n^2 V^n} \, |T(n; pq)|^2$$

and

$$\mathrm{Im}\, T_{\alpha\alpha}(\omega) = -\frac{1}{2} (2\pi)^4 \sum_n \frac{\delta(P_n - p - q)}{N_n^2 V^n} \, |T(n; pq)|^2$$

so that we can identify the forward scattering amplitude $F_{\beta\alpha}(\omega)$ thus

$$F_{\beta\alpha}(\omega) = -\frac{1}{4\pi} T_{\beta\alpha}(\omega). \tag{6.1}$$

Also

$$T_{\beta\alpha}(\omega) = -A_{\beta\alpha}(\omega) - \omega B_{\beta\alpha}(\omega)$$

so that

$$F_{\beta\alpha}(\omega) = \frac{1}{4\pi} \left\{ \delta_{\beta\alpha}(A^{(+)} + \omega B^{(+)}) + \frac{1}{2} [\tau_\beta, \tau_\alpha](A^{(-)} + \omega B^{(-)}) \right\}. \tag{6.2}$$

The dispersion relations are

$$A^{(\pm)}(\omega) = \frac{1}{\pi} \int_m^\infty \frac{d\omega'}{\omega' - \omega} \mathrm{Im}\, A^{(\pm)}(\omega') + \frac{1}{\pi} \int_{-\infty}^{-m} \frac{d\omega'}{\omega' - \omega} \mathrm{Im}\, A^{(\pm)}(\omega') \tag{6.3}$$

$$B^{(\pm)}(\omega) = -\frac{g^2}{m^2 + 2M\omega} \pm \frac{g^2}{m^2 - 2M\omega} + $$

$$+ \frac{1}{\pi} \int_m^\infty \frac{d\omega'}{\omega' - \omega} \mathrm{Im}\, B^{(\pm)}(\omega') + \frac{1}{\pi} \int_{-\infty}^{-m} \frac{d\omega'}{\omega' - \omega} \mathrm{Im}\, B^{(\pm)}(\omega'). \tag{6.4}$$

To convert the integrals over the negative frequency range to integrals over a physical positive frequency range, we make use of a property to be discussed more fully in sec. VII, namely *Crossing symmetry*[12]). Most simply expressed, this property describes that for each Feynman graph of the type (a) there will also exist an identical FEYNMAN graph of the type (b), i. e. with the external lines crossed.

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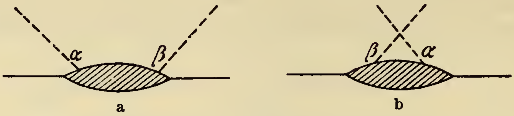
[11]) The reason for this is that the only way we have of handling the imaginary part of the amplitude is by a phase shift expansion. Aside from convergence questions, the real part of the amplitude contains arbitrary functions of the scattering angle, which makes the setting up of a sequence of equations connecting phase shifts impossible. This defect will be remedied in sec. VII and IX.

[12]) This property of quantum field theory was first explicitly stated by Gell-Mann and Goldberger (unpublished).



Thus

$$\lim_{\varepsilon\to 0+} T_{\beta\alpha}(\omega + i\varepsilon) = \lim_{\varepsilon\to 0+} T_{\alpha\beta}(-\omega - i\varepsilon).$$

The consequences of this are that

$$\begin{aligned} \operatorname{Im} A^{(\pm)}(-\omega) &= \mp \operatorname{Im} A^{(\pm)}(\omega) \\ \operatorname{Im} B^{(\pm)}(-\omega) &= \pm \operatorname{Im} B^{(\pm)}(\omega). \end{aligned} \tag{6.5}$$

Hence we get from (6.3) and (6.4), as ω approaches the real axis from above,

$$\begin{aligned} \operatorname{Re} A^{(+)}(\omega) &= \frac{2}{\pi} P \int_m^\infty \omega'\, d\omega' \frac{\operatorname{Im} A^{(+)}(\omega')}{\omega'^2 - \omega^2} \\ \operatorname{Re} A^{(-)}(\omega) &= \frac{2\omega}{\pi} P \int_m^\infty d\omega' \frac{\operatorname{Im} A^{(-)}(\omega')}{\omega'^2 - \omega^2} \end{aligned} \tag{6.6}$$

$$\begin{aligned} \operatorname{Re} B^{(+)}(\omega) &= -\frac{g^2}{M} \frac{\omega}{\omega^2 - (m^2/2M)^2} + \frac{2\omega}{\pi} P \int_m^\infty d\omega' \frac{\operatorname{Im} B^{(+)}(\omega')}{\omega'^2 - \omega^2} \\ \operatorname{Re} B^{(-)}(\omega) &= \frac{g^2}{2M^2} \frac{m^2}{\omega^2 - (m^2/2M)^2} + \frac{2}{\pi} P \int_m^\infty \omega'\, d\omega' \frac{\operatorname{Im} B^{(-)}(\omega')}{\omega'^2 - \omega^2}. \end{aligned} \tag{6.7}$$

In terms of isotopic spin states (appendix B)

$$\begin{aligned} F^{(+)} &= \frac{1}{4\pi} (A^{(+)} + \omega B^{(+)}) = \frac{1}{3} \left(F^{\left(\frac{1}{2}\right)} + 2F^{\left(\frac{3}{2}\right)}\right) \\ F^{(-)} &= \frac{1}{4\pi} (A^{(-)} + \omega B^{(-)}) = \frac{1}{3} \left(F^{\left(\frac{1}{2}\right)} - F^{\left(\frac{3}{2}\right)}\right). \end{aligned} \tag{6.8}$$

We shall however find it more convenient to work in terms of amplitudes for π^+ π^- scattering. There we use the relations

$$\begin{aligned} \frac{1}{2} (F_{\pi^-} - F_{\pi^+}) &= \frac{1}{3} \left(F^{\left(\frac{1}{2}\right)} - F^{\left(\frac{3}{2}\right)}\right) \\ \frac{1}{2} (F_{\pi^-} + F_{\pi^+}) &= \frac{1}{3} \left(F^{\left(\frac{1}{2}\right)} + 2F^{\left(\frac{3}{2}\right)}\right). \end{aligned} \tag{6.9}$$

Combining relations (6.6) and (6.7) we get

$$\mathrm{Re}\,(A^{(-)} + \omega B^{(-)} = \frac{g^2 m^2}{2M^2}\,\frac{\omega}{\omega^2 - (m^2/2M)^2} +$$

$$+ \frac{2\omega}{\pi} P \int_m^\infty d\omega' \,\frac{\mathrm{Im}\,(A^{(-)}(\omega') + \omega' B^{(-)}(\omega'))}{\omega'^2 - \omega^2}$$

i. e.

$$\mathrm{Re}\,\frac{F_{\pi^-}(\omega) - F_{\pi^+}(\omega)}{2} = \frac{g^2}{4\pi}\left(\frac{m}{2M}\right)^2 \frac{2\omega}{\omega^2 - (m^2/2M)^2} +$$

$$+ \frac{\omega}{2\pi^2} P \int_m^\infty \frac{q'\,d\omega'}{\omega'^2 - \omega^2}\,\frac{\sigma_{\pi^-}(\omega') - \sigma_{\pi^+}(\omega')}{2} \tag{6.10}$$

where we have used the optical theorem in the r. h. s. With the conventional notation

$$\frac{g^2}{4\pi}\left(\frac{m}{2M}\right)^2 = f^2;\quad \left(\frac{m^2}{2M}\right)^2 = \omega_{\mathrm{B}}^2$$

we get

$$\frac{1}{2}\mathrm{Re}\,(F_{\pi^-}(\omega) - F_{\pi^+}(\omega)) = \frac{2\omega f^2}{\omega^2 - \omega_{\mathrm{B}}^2} +$$

$$+ \frac{\omega}{2\pi^2} P \int_m^\infty \frac{q'\,d\omega'}{\omega'^2 - \omega^2}\,\frac{\sigma_{\pi^-}(\omega') - \sigma_{\pi^+}(\omega')}{2}. \tag{6.11}$$

This relation only makes sense if the integral on the r. h. s. converges i. e. if $\sigma_{\pi^-}(\omega') - \sigma_{\pi^+}(\omega') \to 0$ faster than $1/\log \omega'$ as $\omega' \to \infty$. The experimental evidence[13]) is that both σ_{π^-} and σ_{π^+} become constant at high energies, and furthermore that the constants are equal. Given the former result the latter must also be true, as will be seen at the end of this section. We assume that the integral converges. It is then possible to use this relation to determine the coupling constant. Following the method of Haber-Schaim[14]) we rewrite eq. (6.11) as follows

$$\frac{1}{2}\,\omega\,\mathrm{Re}\,(F_{\pi^-}(\omega) - F_{\pi^+}(\omega)) - \frac{\omega^4}{4\pi^2} P \int_m^\infty \frac{q'\,d\omega'}{\omega'^2(\omega'^2 - \omega^2)}\,(\sigma_{\pi^-}(\omega') - \sigma_{\pi^+}(\omega')) =$$

$$= 2f^2 + \frac{\omega^2}{2\pi^2} \int_m^\infty \frac{q'\,d\omega'}{\omega'^2}\,\frac{\sigma_{\pi^-}(\omega') - \sigma_{\pi^+}(\omega')}{2} \tag{6.12}$$

(ω_{B}^2 is neglected). The l. h. s. is determined using the known cross sections, measured to about 2 Bev. on one hand, and the low energy angular distribution measurements on the other. If the l. h. s. is plotted against ω^2 and this is extrapolated to $\omega^2 = 0$, one finds that

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[13]) Longo et. al., Phys. Rev. Letters 3, 568 (1959).

[14]) U. Haber-Schaim, Phys. Rev. 104, 1113 (1956).

$$f^2 = 0{,}082 \pm 0{,}015 .$$

A check on the validity of the relation is obtained by comparing the slope of the line, as calculated using the l. h. s. with that obtained on the r. h. s., namely $\frac{1}{4\pi^2}\int_m^\infty \frac{q'\,d\omega'}{\omega'^2}\,(\sigma_{\pi^-}(\omega') - \sigma_{\pi^+}(\omega'))$ and good agreement is obtained.

To combine $A^{(+)}$ and $\omega B^{(+)}$ we rewrite (6.6) thus:

$$\operatorname{Re} A^{(+)}(\omega) = A^{(+)}(0) + \frac{2\omega^2}{\pi} P \int_m^\infty \frac{d\omega'}{\omega'} \frac{\operatorname{Im} A^{(+)}(\omega')}{\omega'^2 - \omega^2}$$

so that we can write

$$\operatorname{Re}\,(A^{(+)}(\omega) + \omega B^{(+)}(\omega)) = A^{(+)}(0) - \frac{g^2}{M}\frac{\omega^2}{\omega^2 - \omega_B^2} + $$
$$+ \frac{2\omega^2}{\pi} P \int_m^\infty \frac{d\omega'}{\omega'} \frac{\operatorname{Im}\,(A^{(+)}(\omega') + \omega' B^{(+)}(\omega'))}{\omega'^2 - \omega^2}.$$

In terms of the π^+ and π^- amplitudes we obtain

$$\frac{1}{2}\operatorname{Re}\,(F_{\pi^-}(\omega) + F_{\pi^+}(\omega)) = \frac{1}{4\pi} A^{(+)}(0) - \frac{g^2}{4\pi}\frac{\omega^2}{M}\frac{1}{\omega^2 - \omega_B^2} +$$
$$+ \frac{\omega^2}{2\pi^2} P \int_m^\infty \frac{q'\,d\omega'}{\omega'} \frac{1}{\omega'^2 - \omega^2} \frac{\sigma_{\pi^-}(\omega') + \sigma_{\pi^+}(\omega')}{2}. \tag{6.13}$$

If σ_{π^+} and σ_{π^-} become constant at high energies, the integral on the r. h. s. converges. This is achieved at the cost of introducing into the formula an unknown "subtraction constant" $1/4\pi \cdot A^{(+)}(0)$. With the given behavior of the total cross sections, the dispersion relation without the subtraction would not converge. The constant can be expressed in terms of the amplitude at threshold. It is easy to see that eq. (6.13) can be rewritten thus:

$$\frac{1}{2}\operatorname{Re}\,(F_{\pi^-}(\omega) + F_{\pi^+}(\omega)) = \frac{1}{2}\operatorname{Re}\,(F_{\pi^-}(m) + F_{\pi^+}(m)) +$$
$$+ \frac{f^2}{M}\frac{q^2}{\omega^2 - \omega_B^2}\frac{1}{1 - \omega_B^2/m^2} + \frac{q^2}{2\pi^2}\int_m^\infty \frac{\omega'\,d\omega'}{q'} \frac{1}{\omega'^2 - \omega^2}\frac{1}{2}\,(\sigma_{\pi^-}(\omega') + \sigma_{\pi^+}(\omega')). \tag{6.14}$$

Given the value of the coupling constant obtained from the first relation, and taking the experimental scattering lengths at threshold[15]), it is possible to check whether this relation agrees with experiment. Within the still large experimental errors, no disagreement with experiment has been found[16]). In practice the difference dispersion relation is also written with a subtraction, which makes the integrals over the cross sections more convergent, and emphasizes the contributions at lower energies. With the neglect of ω_B^2/m^2 the subtracted relation is

$$\frac{1}{2}\operatorname{Re}\left(F_{\pi^-}(\omega) - F_{\pi^+}(\omega)\right) = \frac{\omega}{2m}\operatorname{Re}\left(F_{\pi^-}(m) - F_{\pi^+}(m)\right) - \frac{2f^2q^2}{m^2}\frac{\omega}{\omega^2-\omega_B^2} + $$

$$+ \frac{q^2\omega}{2\pi^2} P\int_m^\infty \frac{d\omega'}{q'}\frac{1}{\omega'^2-\omega^2}\frac{1}{2}\left(\sigma_{\pi^-}(\omega') - \sigma_{\pi^+}(\omega')\right). \tag{6.15}$$

The two dispersion relations can now be combined to give separate relations for the π^+ and π^- amplitudes. Using the conventional notation

$$D_\pm(\omega) = \operatorname{Re} F_{\pi^\pm}(\omega) \tag{6.16}$$

we find that

$$D_-(\omega) = \frac{1}{2}\left(1+\frac{\omega}{m}\right)D_-(m) + \frac{1}{2}\left(1-\frac{\omega}{m}\right)D_+(m) - \frac{2f^2}{m^2}\frac{q^2}{\omega+m^2/2M} +$$

$$+ \frac{q^2}{4\pi^2} P\int_m^\infty \frac{d\omega'}{q'}\left[\frac{\sigma_{\pi^-}(\omega')}{\omega'-\omega} + \frac{\sigma_{\pi^+}(\omega')}{\omega'+\omega}\right] \tag{6.17}$$

and

$$D_+(\omega) = \frac{1}{2}\left(1+\frac{\omega}{m}\right)D_+(m) + \frac{1}{2}\left(1-\frac{\omega}{m}\right)D_-(m) + \frac{2f^2}{m^2}\frac{q^2}{\omega-m^2/2M} +$$

$$+ \frac{q^2}{4\pi^2} P\int_m^\infty \frac{d\omega'}{q'}\left[\frac{\sigma_{\pi^+}(\omega')}{\omega'-\omega} + \frac{\sigma_{\pi^-}(\omega')}{\omega'+\omega}\right]. \tag{6.18}$$

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[15]) At low energies we use the phase shifts α_{2i} ($i = 1/2, 3/2$) for S-wave scattering and $\alpha_{2i,2j}$ $i = 1/2, 3/2, j = 1/2, 3/2$) for P wave scattering, to determine the amplitudes, using

$$\operatorname{Re}(F_{\pi^-} - F_{\pi^+}) \cong \frac{1}{3q}\left(1+\frac{2\omega}{M}+\frac{m^2}{M^2}\right)(\sin 2\alpha_1 + \sin 2\alpha_{11} + 2\sin 2\alpha_{13} - \sin 2\alpha_3 - \sin 2\alpha_{31} - 2\sin 2\alpha_{33})$$

and

$$\operatorname{Re}(F_{\pi^-} + F_{\pi^+}) \cong \frac{1}{3q}\left(1+\frac{2\omega}{M}+\frac{m^2}{M^2}\right)(\sin 2\alpha_1 + \sin 2\alpha_{11} + 2\sin 2\alpha_{13} + 2\sin 2\alpha_3 + 2\sin 2\alpha_{31} + 4\sin 2\alpha_{33})$$

[16]) T. D. SPEARMAN, Nuovo Cim. 15, 147 (1960).

These forms are also useful for the discussion of POMERANCHUK's theorem[17]. The statement of the theorem is that if

$$\lim_{\omega=\infty} \sigma_{\pi^+}(\omega) = \text{constant} = \sigma_+$$
$$\lim_{\omega=\infty} \sigma_{\pi^-}(\omega) = \text{constant} = \sigma_- \tag{6.19}$$

then

$$\sigma_+ = \sigma_- . \tag{6.20}$$

To demonstrate this, we assume that the limits are not equal, so that if

$$\sigma_{\pi^+}(\omega) - \sigma_{\pi^-}(\omega) = \delta\sigma(\omega)$$

then

$$\lim_{\omega=\infty} \delta\sigma(\omega) = \delta\sigma \neq 0 .$$

Now in (6.18) we may examine the high energy behavior of $D_+(\omega)$. For $\omega \gg \Omega$ where Ω is an energy such that beyond it the cross sections are effectively constant, we have

$$\begin{aligned} D_+(\omega) \sim \frac{\omega}{2m}\,(D_+(m) - D_-(m)) \; + \\ + \frac{2f^2}{m^2}\,\omega + \frac{\omega^2}{4\pi^2} P\int_m^{\Omega} \frac{d\omega'}{q'}\,\frac{2\omega'}{\omega'^2-\omega^2}\,\sigma_{\pi^-}(\omega') + \frac{\omega^2\sigma_-}{4\pi^2} P\int_{\Omega}^{\infty} \frac{d\omega'}{q'}\,\frac{2\omega'}{\omega'^2-\omega^2} \; + \\ + \frac{\omega^2}{4\pi^2} P\int_m^{\Omega} \frac{d\omega'}{q'}\,\frac{\delta\sigma(\omega')}{\omega'+\omega} + \frac{\omega^2\,\delta\sigma}{4\pi^2}\int_{\Omega}^{\infty} \frac{d\omega'}{q'}\,\frac{1}{\omega'+\omega} . \end{aligned} \tag{6.21}$$

The leading term is the last one, and we find that for $\omega \gg \Omega$

$$D_+(\omega) \sim \frac{\delta\sigma}{4\pi^2}\,\omega \log \frac{\omega}{\Omega} .$$

Thus we have

$$\operatorname{Re} F_{\pi^+}(\omega) \sim \frac{\delta\sigma}{4\pi^2}\,\omega \log\frac{\omega}{m} + O(\omega)$$
$$\operatorname{Im} F_{\pi^+}(\omega) \sim \frac{\omega}{4\pi}\,\sigma_+ . \tag{6.22}$$

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[17] I. IA. POMERANCHUK, Soviet Physics JETP 34, 499 (1958).

If we write the forward scattering amplitude in terms of the complex phase shifts $\delta_l + i\eta_l$, $\eta_l \geq 0$, we have

$$\begin{aligned} \operatorname{Im} F_{\pi^+}(\omega) &= \sum_l (2l+1)\left(\frac{1}{2q} - \cos 2\delta_l \frac{e^{-2\eta_l}}{2q}\right) \\ \operatorname{Re} F_{\pi^+}(\omega) &= \sum_l (2l+1) \sin 2\delta_l \frac{e^{-2\eta_l}}{2q}. \end{aligned} \tag{6.23}$$

Now in order to get a constant cross section at infinity, we must assume that the target has a finite "effective radius" R, i. e. that the scattering takes place in states with $l \leq L \sim \omega R$. Then

$$\begin{aligned} \operatorname{Im} F_{\pi^+}(\omega) &\approx \frac{\omega}{2} R^2 - \sum_{l=0}^{\omega R}(2l+1)\cos 2\delta_l \frac{e^{-2\eta_l}}{2q} \\ \operatorname{Re} F_{\pi^+}(\omega) &\approx \sum_{l=0}^{\omega R} (2l+1) \sin 2\delta_l \frac{e^{-2\eta_l}}{2q} \end{aligned} \tag{6.24}$$

and the second term in $\operatorname{Im} F_{\pi^+}(\omega)$ will be smaller than the leading term because of (*i*) absorption and (*ii*) incoherence of the many phase shifts. For the same reason, any reasonable model gives the results that (*a*) the real part of the amplitude will always be smaller than the imaginary part at high energies, and (*b*) the real part of the amplitude can at best increase as ω. Hence

$$\delta\sigma = 0. \tag{6.25}$$

This completes our discussion of the one-dimensional dispersion relations. These relations have been, and still are exceedingly useful in the correlation of data, in the elimination of spurious solutions in phase shift analyses, and they will undoubtedly play an important role in the analysis of K meson scattering. We do not discuss these applications in more detail, but rather go on to a discussion of the Mandelstam representation, which provides a generalization of dispersion relations, with the help of which one may hope to construct a useful low energy approximation scheme.

VII. The Mandelstam Representation

The dispersion relations in the s variable with t fixed, discussed in sec. V turned out to be unmanageable because the behavior of the amplitude as a function of t was completely unknown. Mandelstam observed that in the related process $\pi + \pi \rightarrow \mathrm{N} + \overline{\mathrm{N}}$ the t variable played the role of an (energy)^2, and that the structure of the matrix element for this process (as seen in a reduction formula) suggested that for s fixed, a dispersion relation in the t variable might also exist[18].

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[18] With the use of local commutativity and the mass spectrum alone, but without the use of the unitarity condition, it is not possible to prove such a dispersion relation. If such a relation could be proved, it could presumably be used to prove the validity of the Mandelstam representation in a certain domain. For meson-meson scattering, where the same analyticity properties hold for t variable and s fixed as for s variable and t fixed, Mandelstam has been able to prove analyticity in the product of the cut s and t planes in the domain $|s\,t\,u| < 324 m^6$. (Nuovo Cimento **15**, 658 (1960)).

Making the simplest possible assumption about the behavior of the amplitude when both s and t are allowed to become complex, **Mandelstam** was led to the following representations for the meson-nucleon scattering amplitude[19]:

$$A^{(\pm)}(s,t,u) = \frac{1}{\pi^2}\int\limits_{(M+m)^2}^{\infty} ds' \int\limits_{4m^2}^{\infty} dt' \, \frac{A_{13}^{(\pm)}(s't')}{(s'-s)(t'-t)} +$$

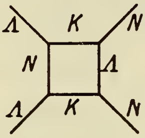
$$+ \frac{1}{\pi^2}\int\limits_{(M+m)^2}^{\infty} du' \int\limits_{4m^2}^{\infty} dt' \, \frac{A_{23}^{(\pm)}(u't')}{(u'-u)(t'-t)} + \frac{1}{\pi^2}\int\limits_{(M+m)^2}^{\infty} ds' \int\limits_{(M+m^2)}^{\infty} du' \, \frac{A_{12}^{(\pm)}(s'u')}{(s'-s)(u'-u)} \tag{7.1}$$

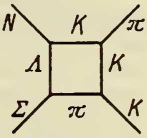
$$B^{(\pm)}(s,t,u) = \frac{g^2}{M^2-s} \mp \frac{g^2}{M^2-u} + \frac{1}{\pi^2}\int\limits_{(M+m)^2}^{\infty} ds' \int\limits_{4m^2}^{\infty} dt' \, \frac{B_{13}^{(\pm)}(s't')}{(s'-s)(t'-t)} +$$

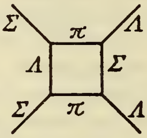
$$+ \frac{1}{\pi^2}\int\limits_{(M+m)^2}^{\infty} du' \int\limits_{4m^2}^{\infty} dt' \, \frac{B_{23}^{(\pm)}(u't')}{(u'-u)(t'-t)} + \frac{1}{\pi^2}\int\limits_{(M+m)^2}^{\infty} ds' \int\limits_{(M+m)^2}^{\infty} du' \, \frac{B_{12}^{(\pm)}(s'u')}{(s'-s)(u'-u)} \tag{7.2}$$

The limits of integration are determined by the unitarity condition (sec. IV). The spectral functions can also be determined with help of the unitarity condition[19]. As will be seen later, it may not be necessary to calculate the spectral functions for low energy applications, and we do not, in fact, discuss the procedure in this paper.

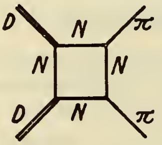
There are known cases for which the conjecture has been disproved in fourth order perturbation theory. In the following examples,

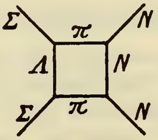


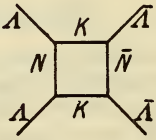




the cuts extend beyond those conjectured by **Mandelstam**, and for others, examples of which are given below,







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the conjecture is just false. In all of the counterexamples, scattering of particles which may be viewed as "composite"[20] is involved, and it may be that the conjecture is only correct for the lightest bosons and fermions in the theory, viz. mesons and nucleons. It may also only be valid in a restricted domain, which would not necessarily be too serious a restriction, as the applications of the representation are to the low energy processes. For the time being, we shall discuss some of the properties of the representation on the assumption that it is correct for all s, t, u.

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[19] S. **Mandelstam**, Phys. Rev. **112**, 1344 (1958).

[20] In the sense of R. **Oehme**, Nuovo Cimento **13**, 778 (1959). In this paper a quantitative distinction between "simple" and "composite" particles is proposed.

a) Generalized Crossing Symmetry

The "symmetric" appearance of the three invariants s, t, u takes account of the property which we may call "generalized crossing symmetry". We discussed this property briefly when we treated forward scattering in sec. VI. As mentioned before, the symmetry is a consequence of what happens when two of the external lines in a general graph are interchanged or crossed. If the crossed graph represents the same process as the uncrossed one, we obtain a symmetry property. Thus in pion-nucleon scattering, the interchange $(q, \alpha) \leftrightarrow (-q', \beta)$ i. e. $s \leftrightarrow u$ leads to the same process. It is important to remember that this statement has physical significance only if the amplitude has analyticity properties such that the change of the first variable in $T(s, t, u)$ to a value "u" in a different region, and the simultaneous change of the third variable to the value "s" can be carried out by analytic continuation. In general such analyticity properties have only been proved in a few cases (for example in the one-dimensional dispersion relations of sec. V), but they are contained in the MANDELSTAM representation. The crossing symmetry leads to relations between the spectral functions. We want

$$T(p'q', \beta; pq, \alpha) = \bar{u}(p') \left\{ \left[A^{(+)}(s, t, u)\, \delta_{\beta\alpha} + \frac{1}{2}\, [\tau_\beta, \tau_\alpha]\, A^{(-)}(s, t, u) \right] \right.$$

$$\left. + i\gamma\, \frac{q + q'}{2} \left[B^{(+)}(s, t, u)\, \delta_{\beta\alpha} + \frac{1}{2} [\tau_\beta, \tau_\alpha]\, B^{(-)}(s, t, u) \right] \right\}$$

to be invariant under the interchange $(q, \alpha) \leftrightarrow (-q', \beta)$ [21]. This implies that

$$\begin{aligned} A^{(\pm)}(s, t, u) &= \pm A^{(\pm)}(u, t, s) \\ B^{(\pm)}(s, t, u) &= \mp B^{(\pm)}(u, t, s). \end{aligned} \tag{7.3}$$

Consequently

$$\begin{aligned} A_{13}^{(\pm)}(x, y) &= \pm A_{23}^{(\pm)}(x, y) \\ A_{12}^{(\pm)}(x, y) &= \pm A_{12}^{(\pm)}(y, x) \end{aligned} \tag{7.4a}$$

and

$$\begin{aligned} B_{13}^{(\pm)}(x, y) &= \mp B_{23}^{(\pm)}(x, y) \\ B_{12}^{(\pm)}(x, y) &= \mp B_{12}^{(\pm)}(y, x). \end{aligned} \tag{7.4b}$$

---

[21] This symmetry has a particularly simple form in the Breit system for which

$$s = M^2 + m^2 + \frac{1}{2}\, \Delta^2 + 2E\omega; \quad u = M^2 + m^2 + \frac{1}{2}\, \Delta^2 - 2E\omega.$$

Then

$$\lim_{\varepsilon \to 0+} A^{(\pm)}(\omega + i\varepsilon, \Delta) = \pm \lim_{\varepsilon \to 0+} A^{(\pm)}(-\omega - i\varepsilon, \Delta)$$

i. e.

$$A^{(\pm)}(\omega, \Delta) = \pm \left(A^{(\pm)}(-\omega, \Delta)\right)^*; \quad B^{(\pm)}(\omega, \Delta) = \mp \left(B^{(\pm)}(-\omega, \Delta)\right)^* \text{ for } \Delta \text{ real.}$$

If the crossed process is not the same as the original one, e. g. $K^+ + P \to K^+ + P$ then we no longer have a *symmetry*, but rather a relation between the amplitudes for two entire different processes. In our example, we would find a relation between the K^+P and K^-P scattering amplitudes, and this would imply the need for studying the two processes together. For pion-nucleon scattering, the interchange $s \leftrightarrow t$ leads to a different process, namely $\pi + \pi \to N + \overline{N}$, and as we will see, this process has to be studied simultaneously with the scattering process. Incidentally, if we look at the process $\pi + \pi \to N + \overline{N}$, the ordinary crossing symmetry (interchange of the two mesons) is just the symmetry connected with the boson character of the pions[22]). When we study pion-pion scattering, *all* crossed processes are scattering processes, and the general crossing symmetry gives us much more information.

b) One Dimensional Dispersion Relations

It is clear that the one-dimensional dispersion relations must be contained in the general representation. To see this in detail, and in particular to see the connection between the absorptive parts, which are given by unitarity, and the spectral functions, we proceed as follows: Take t real and fixed, with $t < 4m^2$ so that $t' - t$ is always positive. Writing

$$\frac{1}{(s'-s)\,(u'-u)} = \left(\frac{1}{s'-s} + \frac{1}{u'-u}\right)\frac{1}{s'+u'-s-u} =$$

$$= \left(\frac{1}{s'-s} + \frac{1}{u'-u}\right)\frac{1}{u'+s'-2M^2-2m^2+t}$$

since $u + s = 2M^2 + 2m^2 - t$, we can rewrite eq. (7.1) thus

$$A^{(\pm)}(s,t,u) =$$

$$= \frac{1}{\pi^2}\int\limits_{(M+m)^2}^{\infty}\frac{ds'}{s'-s}\left(\int\limits_{4m^2}^{\infty}dt'\,\frac{A_{13}^{(\pm)}(s'\,t')}{t'-t} + \int\limits_{(M+m)^2}^{\infty}du'\,\frac{A_{12}^{(\pm)}(s',u')}{s'+u'-2M^2-2m^2+t}\right) +$$

$$+ \frac{1}{\pi^2}\int\limits_{(M+m)^2}^{\infty}\frac{du'}{u'-u}\left(\int\limits_{4m^2}^{\infty}dt'\,\frac{A_{23}^{(\pm)}(u't')}{t'-t} + \int\limits_{(M+m)^2}^{\infty}ds'\,\frac{A_{12}^{(\pm)}(s',u')}{s'+u'-2M^2-2m^2+t}\right).$$

In the second integral we introduce the new variable t' via $u' = 2M^2 + 2m^2 - s' - t'$ and in the fourth we introduce t' via $s' = 2M^2 + 2m^2 - u' - t'$.
The above equation may now be rewritten in the form

$$A^{(\pm)}(s,t,u) = \frac{1}{\pi}\int\limits_{(M+m)^2}^{\infty}ds'\,\frac{A_{\mathrm{I}}^{(\pm)}(s',t)}{s'-s} + \frac{1}{\pi}\int\limits_{(M+m)^2}^{\infty}du'\,\frac{A_{\mathrm{II}}^{(\pm)}(u',t)}{u'-u} \qquad (7.5)$$

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[22]) This is the substance of the discussion of crossing symmetry given by Feldman and Matthews, Phys. Rev. 102, 1421 (1956).

with

$$A_{\rm I}^{(\pm)}(s',t) = \frac{1}{\pi}\int\limits_{4m^2}^{\infty} dt'\, \frac{A_{13}^{(\pm)}(s',t')}{t'-t} - \frac{1}{\pi}\int\limits_{-\infty}^{(M-m)^2-s'} dt'\, \frac{A_{12}^{(\pm)}(s',u'(t'))}{t'-t} \tag{7.6a}$$

$$A_{\rm II}^{(\pm)}(u',t) = \frac{1}{\pi}\int\limits_{4m^2}^{\infty} dt'\, \frac{A_{23}^{(\pm)}(u't')}{t'-t} - \frac{1}{\pi}\int\limits_{-\infty}^{(M-m)^2-u'} dt'\, \frac{A_{12}^{(\pm)}(s'(t'),u')}{t'-t}. \tag{7.6b}$$

It can easily be seen that this is just the form of the one-dimensional dispersion relation (5.18). The absorptive parts $A_{\rm I}^{(\pm)}$, $A_{\rm II}^{(\pm)}$ will be real provided that

$$-4mM < t < 4m^2 \tag{7.7}$$

because t' lies in the domains $(-\infty, -4mM)$ and $(4m^2, \infty)$. The same forms, aside from the one-particle poles, hold for $B^{(\pm)}(s,t,u)$.

c) Subtractions

The **Mandelstam** representation was written down on the assumption that all the integrals in it converge. In general this need not be true, and we may have to make subtractions. Thus if (say) $A_{12}^{(\pm)}(s',u')$ does not vanish as $u' \to \infty$ we may have to write instead of

$$\frac{1}{\pi^2}\iint ds'\, du' \frac{A_{12}^{(\pm)}(s'u')}{(s'-s)\,(u'-u)}$$

the expression

$$\frac{u-u_0}{\pi^2}\iint ds'\, du' \frac{A_{12}^{(\pm)}(s'u')}{(s'-s)\,(u'-u)\,(u'-u_0)} + \frac{1}{\pi}\int ds'\, \frac{f_0(s')}{s'-s}.$$

If one more subtraction were necessary in the u' variable, we would have

$$\frac{(u-u_0)}{\pi^2}\iint ds'\, du' \frac{A_{12}^{(\pm)}(s',u')}{(s'-s)\,(u'-u_0)}\left(\frac{1}{u'-u} - \frac{1}{u'-u_1}\right) +$$

$$+\frac{1}{\pi}\int ds'\, \frac{f_0(s')}{s'-s} + \frac{u-u_0}{\pi}\int ds'\, \frac{f_1(s')}{s'-s}.$$

If now in this form an s' subtraction were necessary, we would get

$$\frac{(u-u_0)\,(u-u_1)\,(s-s_0)}{\pi^2}\iint ds'\, du' \frac{A_{12}^{(\pm)}(s'u')}{(s'-s)\,(s'-s_0)\,(u'-u)\,(u'-u_0)\,(u'-u_1)} +$$

$$+\frac{(u-u_0)\,(u-u_1)}{\pi}\int du'\, \frac{g_0(u')}{u'-u} + \frac{s-s_0}{\pi}\int ds'\, \frac{f_0(s')}{(s'-s)\,(s'-s_0)} +$$

$$+\frac{(u-u_0)\,(s-s_0)}{\pi}\int ds'\, \frac{f_1(s')}{(s'-s)\,(s'-s_0)} + \alpha u + \beta.$$

We thus see the general pattern: we end up with (*i*) double dispersion integrals, (*ii*) single dispersion integrals and (*iii*) overall subtraction terms. The last will be polynomials in s, t, u, and will appear *when the single dispersion integrals need subtractions.* Now if we express the invariants in terms of the center of mass variables

$$\begin{aligned} s &= (E+\omega)^2 = W^2 \\ t &= -2q^2(1-\cos\theta) \qquad (7.8) \\ u &= 2M^2 + 2m^2 - W^2 + 2q^2(1-\cos\theta) \end{aligned}$$

then terms like

$$\frac{1}{\pi}\int ds'\,\frac{f_0(s')}{s'-s} \quad\text{and}\quad \frac{s-s_0}{\pi}\int ds'\,\frac{f_0(s')}{(s'-s)(s'-s_0)}$$

are independent of u or t, i. e. independent of $\cos\theta$. This means that these terms contribute to the S-wave amplitude only, and $f_0(s)$ only contributes to the absorptive part of the S-wave amplitude. Similarly

$$\frac{u-u_0}{\pi}\int ds'\,\frac{f_1(s')}{s'-s} \quad\text{and}\quad \frac{(u-u_0)(s-s_0)}{\pi}\int ds'\,\frac{f_1(s')}{(s'-s)(s'-s_0)}$$

only give contributions to the S and P wave amplitudes. Because of the strict limitations imposed by unitarity on the individual partial waves, the high energy behavior of $f_0(s)$ and $f_1(s)$ cannot be too singular. The unitarity limitations follow from the fact that in the center of mass system, the relation between the T function and the scattering amplitude is

$$T = (\text{const.})\,W\,F(W,\cos\theta). \qquad (7.9)$$

For pion-pion scattering, for example

$$F(W,\cos\theta) = \sum_{l=0}^{\infty}(2l+1)\,f_l(W)\,P_l(\cos\theta) \qquad (7.10)$$

with

$$f_l(W) = \frac{e^{2i\delta_l(W)}\,e^{-2\eta_l(W)}-1}{2iq} \qquad (7.11)$$

($\delta_l + i\eta_l$ ($\eta_l > 0$) is the general complex phase shift). Thus for a particular angular momentum state, at high energies

$$T_l(W) \sim (\text{const.})\,[-8\pi i + 8\pi i\,e^{2i\delta_l(W)}\,e^{-2\eta_l(W)}]. \qquad (7.12)$$

In the limit of large absorption

$$T_l(W) \sim (\text{const.})\,(-8\pi i)$$

and $T_l(W)$ cannot increase more rapidly than this. For meson-nucleon scattering a similar dimensional argument shows that any partial wave amplitude must go to zero at high energies. Thus the single dispersion integrals can have at most one subtraction (for boson-boson scattering) and the general form of the MANDELSTAM representation can thus have at most one unknown subtraction constant. As we shall see, the single dispersion integrands can be calculated with the use of the unitarity condition. Thus the symmetric treatment of all channels in a given process allows us to avoid the difficulty inherent in the one-dimensional dispersion relations, that the subtractions were unknown functions of one of the invariants. Here the analyticity properties of these subtraction terms are known, and their relation to the amplitude for another channel is explicitly seen.

We conclude this section by presenting a slight reformulation of the subtraction procedure[23]), which will turn out to be useful in later applications. Instead of writing

$$\frac{1}{t'-t} \rightarrow \frac{t-t_0}{(t'-t)\,(t'-t_0)} + \frac{1}{t'-t_0}$$

we write

$$\frac{1}{t'-t} \rightarrow \left(\frac{1}{t'-t} - \frac{1}{2} \int_{-1}^{1} \frac{d(\cos\theta(s,t))}{t'-t} + \frac{1}{2} \int_{-1}^{1} \frac{d(\cos\theta(s,t))}{t'-t} \right) \tag{7.13}$$

which leads to a subtraction of the S-wave absorptive part. We have

$$\cos\theta(s,t) = 1 + \frac{t}{2q^2}$$

with

$$4q^2 = s - 2(M^2+m^2) + \frac{(M^2-m^2)^2}{s}.$$

Hence

$$\frac{1}{2} \int_{-1}^{1} \frac{d(\cos\theta(s,t))}{t'-t} = \frac{1}{4q^2} \int_{-4q^2}^{0} \frac{dt}{t'-t} = \frac{1}{t_u(s)-t_l(s)} \int_{t_L}^{t_u} \frac{dt'}{t'-t} =$$

$$= \frac{1}{t_u(s)-t_l(s)} \log \frac{t'-t_L(s)}{t'-t_u(s)} \equiv l_I^{(0)}(t',s) \tag{7.14}$$

where we have written t_u and t_l for the upper and lower limits in the t-integration. Thus

$$\frac{1}{t'-t} \rightarrow \left(\frac{1}{t'-t} - l_I^{(0)}(t',s) \right) + \frac{1}{2} \int_{-1}^{1} \frac{d(\cos\theta(s,t))}{t'-t}.$$

As $t' \rightarrow \infty$

$$\frac{1}{t'-t} - l_I^{(0)}(t',s) \rightarrow \frac{t - \frac{1}{2}\,(t_u+t_l)}{t'^2} + O\left(\frac{1}{t'^3}\right) \tag{7.15}$$

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[23]) S. MANDELSTAM, Phys. Rev. **115**, 1741 (1959).

so that this is a proper subtraction. If we need to, *or wish to,* make another subtraction, we may write

$$\begin{aligned}\frac{1}{t'-t} \to &\left(\frac{1}{t'-t} - l_{\mathrm{I}}^{(0)}(t',s) - 3\cos\theta(s,t)\, l_{\mathrm{I}}^{(1)}(t',s)\right) + \\ &+ \frac{1}{2}\int_{-1}^{1}\frac{d(\cos\theta(s,t))}{t'-t} + \frac{3}{2}\cos\theta(s,t)\int_{-1}^{1}\frac{\cos\theta(s,t)\, d(\cos\theta(s,t))}{t'-t}\end{aligned} \tag{7.16}$$

where

$$l_{\mathrm{I}}^{(1)}(t',s) = \frac{1}{2}\int_{-4q^2}^{0}\frac{dt}{2q^2}\,\frac{\left(1+\dfrac{t}{2q^2}\right)}{t'-t}. \tag{7.17}$$

Since as $t' \to \infty$

$$\begin{aligned}3\, l_{\mathrm{I}}^{(1)}(t',s)\cos\theta(s,t) &= \frac{3}{2}\left(1+\frac{t}{2q^2}\right)\int_{-4q^2}^{0}\frac{dt''}{2q^2}\,\frac{\left(1+\dfrac{t''}{2\,q^2}\right)}{t'-t''} \approx \\ &\approx \frac{t-\dfrac{1}{2}(t_{\mathrm{u}}+t_{\mathrm{L}})}{t'^{2}} + O\left(\frac{1}{t'^{3}}\right)\end{aligned} \tag{7.18}$$

we have

$$\frac{1}{t'-t} - l_{\mathrm{I}}^{(0)}(t',s) - 3\cos\theta(s,t)\, l_{\mathrm{I}}^{(1)}(t',s) = O\left(\frac{1}{t'^{3}}\right) \tag{7.19}$$

i. e. this is again a proper subtraction.
Thus the subtracted form of a term like

$$\frac{1}{\pi^2}\iint ds'\, dt'\,\frac{A_{13}(s',t')}{(s'-s)\,(t'-t)}$$

will be

$$\begin{aligned}&\frac{1}{\pi^2}\iint ds'\, dt'\,\frac{A_{13}(s',t')}{(s'-s)}\left\{\frac{1}{t'-t} - l_{\mathrm{I}}^{(0)}(t',s') - 3\cos\theta(s',t')\, l_{\mathrm{I}}^{(1)}(t',s')\right\} + \\ &+ \frac{1}{\pi^2}\int ds'\,\frac{1}{s'-s}\left\{\frac{1}{2}\int_{-1}^{1} d(\cos\theta(s',t))\int\frac{A_{13}(s',t')}{t'-t}\,dt' + \right. \\ &\left. + \frac{3}{2}\cos\theta(s',t)\int_{-1}^{1}\cos\theta(s',t)\, d(\cos\theta(s',t))\int\frac{A_{13}(s'\,t')}{t'-t}\,dt'\right\}\end{aligned}$$

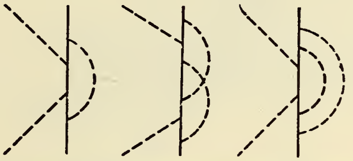
and similarly for the orther terms in the representation. The second term has the form

$$\frac{1}{\pi}\int d s' \, \frac{1}{s'-s}\{A_{\rm I}^{(0)}(s') + 3\cos\theta\,(s',t)\, A_{\rm I}^{(1)}(s')\}.$$

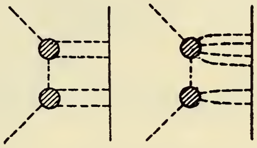
In this way the separation of the absorptive parts of the first n (in general) partial wave amplitudes is made more explicit.

d) The Spectral Functions

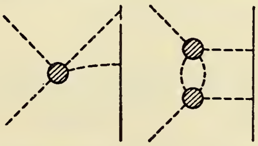
In this section we discuss the regions in which the spectral functions do not vanish. A qualitative idea of the domains can be obtained by looking at Feynman graphs for the processes[24]). Consider for example pion-nucleon scattering: the graphs which do not take into account any meson-meson interaction (direct or through closed loops), of which the following are examples



show that whereas the "mass" of the intermediate state in channel I (read upwards) will start at $(M+m)$, the "mass" of the intermediate state for channel III (read from left to right on the graph) starts at $2M$. If we consider graphs in which the meson-meson interaction is included, we see that the graphs divide into two classes: in those in which the intermediate "mass" for channel I starts at $(M+m)$, for which the "mass" of the intermediate state for channel III starts at $4m$



and those in which the "mass" of the intermediate state for channel III starts at $2m$, but for which the intermediate "mass" in channel I starts at $(M+2m)$.



The reason for this is the absence of a three-pion vertex, which follows from the assumed invariance of the interaction under the G operation[25]). The same arguments hold for a comparison of channel II and III. Thus a plot of, say, $A_{13}^{(\pm)}$ would look like what is shown in fig. 3a. For the same reason the weight function in the Mandelstam representation for pion-pion scattering cannot reach both lower limits of integration ($4m^2$) simultaneously. From the two types of graphs

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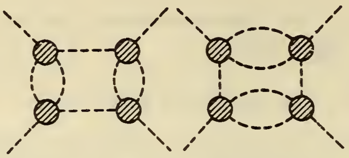
[24]) Cini and Fubini, Annals of Physics 10, 352 (1960).

[25]) The G operation is a product of charge conjugation and a 180° rotation about the y axis in i-spin space, and it has the property that $G|n\rangle = (-1)^n|n\rangle$ for an n-pion state.



we see that if the intermediate "mass" in channel I can become as small as $2m$ then that for channel III cannot be smaller than $4m$ etc. The boundary for this

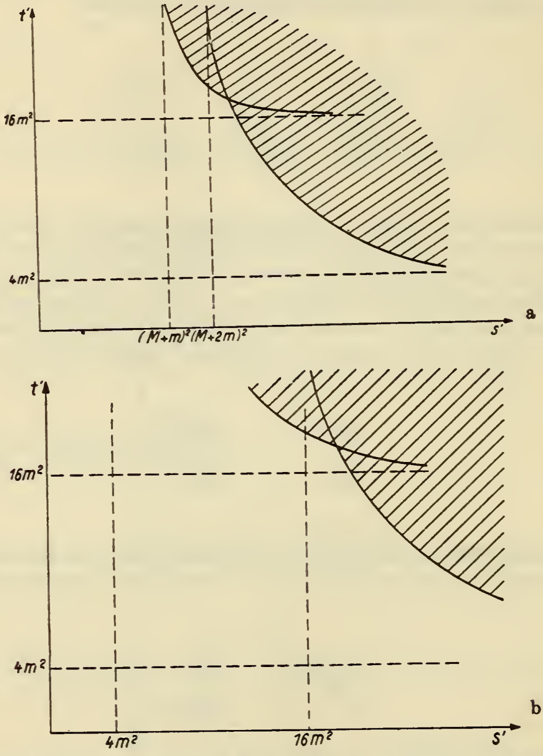


Fig. 3. Regions of nonvanishing spectral functions for meson-nucleon and meson-meson scattering.

process is plotted in fig. 3b. In both cases the boundaries can be calculated in perturbation theory[26]). In both cases the spectral functions are non-zero in a region of the invariant variables in which inelastic processes for the energies of channel I or channel III can occur. This means that, a priori, a calculation of the spectral functions in a certain region of the *s-t* plane (or the *u-t* plane) must include contributions from inelastic intermediate processes. It may be, of course, that this is not necessary.

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[26]) S. Mandelstam, Phys. Rev. 115, 1752 (1959).

The fact that the spectral functions vanish in the elastic region (for all three processes) is the basic of an approximation method discussed under the heading "Single Dispersion Integral Method" in section VIII. This method has been discussed by CINI and FUBINI[24]).

e) Individual Partial Wave Dispersion Relations

In the discussion of the absorptive parts of the individual partial waves, one can make use of another property of the MANDELSTAM representation[27]), namely the explicitness of the dependence on $\cos\theta$. This allows one to project out the individual partial wave amplitudes, and to examine their analyticity properties. These amplitudes are interesting because in the low energy region, before the onset of inelastic processes, the unitarity condition takes a particularly simple form. Consider process I: in terms of the center of mass variables appropriate to it (7.8) the MANDELSTAM representation is

$$A^{(\pm)} = \frac{1}{\pi^2}\iint ds'\,dt'\,A_{13}^{(\pm)}(s't')\,\frac{1}{s'-W^2}\,\frac{1}{t'+2q^2(1-\cos\theta)} +$$

$$+\frac{1}{\pi^2}\iint du'\,dt'\,A_{23}^{(\pm)}(u't')\,\frac{1}{u'-(E-\omega)^2+2q^2(1+\cos\theta)}\,\frac{1}{t'+2q^2(1-\cos\theta)} +$$

$$+\frac{1}{\pi^2}\iint ds'\,du'\,A_{12}^{(\pm)}(s'u')\,\frac{1}{s'-W^2}\,\frac{1}{u'-(E-\omega)^2+2q^2(1+\cos\theta)}.$$

Thus

$$A_l^{(\pm)}(q^2) = \frac{1}{2}\int_{-1}^{1} d(\cos\theta)\,A^{(\pm)}(q^2,\cos\theta)\,P_l(\cos\theta) \tag{7.20}$$

has the representation

$$A_l^{(\pm)}(q^2) = \frac{1}{\pi^2}\iint ds'\,dt'A_{13}^{(\pm)}(s't')\,\frac{1}{s'-W^2}\,\frac{1}{2q^2}\,Q_l\left(1+\frac{t'}{2q^2}\right) +$$

$$+\frac{1}{\pi^2}\iint du'\,dt'\,A_{23}^{(\pm)}(u't')\,\frac{1}{2q^2}\,\frac{1}{u'+t'+4q^2-(E-\omega)^2}\left\{Q_l\left(\frac{(E-\omega)^2-u'}{2q^2}-1\right)-\right.$$

$$\left.-\,Q_l\left(1+\frac{t'}{2q^2}\right)\right\} +$$

$$+\frac{1}{\pi^2}\iint ds'\,du'\,A_{12}^{(\pm)}(s'u')\,\frac{1}{s'-W^2}\left(-\frac{1}{2q^2}\right)Q_l\left(\frac{(E-\omega)^2-u'}{2q^2}-1\right). \tag{7.21}$$

---

[27]) See "The Effective Range Approach" in the next section.

We may use this representation to examine the singularities in the q^2 plane. In the first term we have a cut for $(E + \omega)^2 = W^2 \geq (M + m)^2$, i. e. for $q^2 \geq 0$. The factor $Q_l(1 + t'/2q^2)$ gives rise to singularities when

$$1 + \frac{t'}{2q^2} = \pm 1$$

i. e. $t' = 0$ or $t' = -4q^2$. Since $t' \geq 4m^2$ we get a branch cut extending from $q^2 = -m^2$ to $-\infty$. In the second term, the factor $[u' + t' + 4q^2 - (E - \omega)^2]^{-1}$ does not give rise to any singularities. The only additional branch cut is one occuring in the second and third terms when $(E - \omega)^2 - u' - 2q^2 = \pm 2q^2$ i. e. when

$$4q^2 = \frac{[(M + m)^2 - u'] [(M - m)^2 - u']}{M^2 + m^2 \pm (M^2 + m^2 - u')} .$$

(*i*) the lower sign solution yields a cut extending from $q^2 = 0$ to $+\infty$. (*ii*) the upper sign solution yields a cut from $-\infty$ to $-M^2$. In the $B_l^{\pm}$ equation, the integral terms have the same singularities. The pole at $s = M^2$ gives rise to an s-wave pole at $q^2 = -m^2 + (m^2/2M)^2$ and the term $1/M^2 - u$ gets spread into a cut extending from $-m^2 + (m^2/2M)^2$ to $-m^2 + qm^4/(4M^2 + 8m^2)$. Note that as $M \rightarrow \infty$, this cut contracts into a pole at $q^2 = -m^2$. This is the pole in the $(^3/_2, ^3/_2)$ amplitude in the static theory. It is not a "bound state pole", as there are no single particle terms in this amplitude. The only bona fide single particle terms can only occur in the $(^1/_2, ^1/_2)$ amplitude.

In pion-pion scattering the same can be done. There the individual partial waves have the form

$$f_l(q) = \frac{e^{i\delta_l} \sin \delta_l}{q}$$

with δ_l real below the threshold for inelastic processes, so that the unitarity condition takes the simple form

$$\operatorname{Im} \frac{1}{f_l(q)} = -q.$$

For meson-nucleon scattering, the relation between the functions

$$f_l^{(\pm)} = \frac{e^{i\delta_l^{(\pm)}} \sin \delta_l^{(\pm)}}{q}$$

and the functions $A_l^{\pm}(q)$, $B_l^{\pm}(q)$ which have simple analyticity properties is complicated by kinematical factors which give rise to additional branch points[28]). To illustrate the approximation technique based on dispersion relations for single partial wave amplitudes, it will be simpler to consider pion-pion scattering. There is a more valid reason for discussing this problem first: as can be seen from the general crossing aspects, it is the only problem which can be treated independently of all others within the framework of the two-particle approximation to unitarity which we shall make, and its solution is in principle necessary for the solution of the problem, which in turn is coupled to the meson-nucleon scattering amplitude. Furthermore it is a problem in which, at first sight, the approximations made are most valid.

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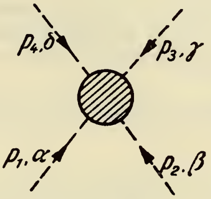
[28]) See S. W. MacDowell, Phys. Rev. 116, 774 (1959).

VIII. Pion-Pion Scattering

In this section we shall discuss the application of the approximation techniques suggested by the previously discussed properties of the Mandelstam representation. In both methods to be discussed, the spirit of the approximation is to satisfy the analyticity properties suggested by the representation, and the crossing symmetries, exactly, and to satisfy the unitarity condition approximately. This is the substance of the Chew-Low treatment of the static model for the meson-nucleon system[29], and the success of this approach there leads to the hope that the relativistic generalization will also be successful. The two approaches to be discussed give the same equations for pion-pion scattering[24], and exhibit only technical differences. One or the other may however be more useful for more complicated problems. The first one to be discussed is the "effective range approach" of Chew and Mandelstam[30]. The other approach consists of making a certain number of subtractions, presumably a number large enough to make the approximation self-consistent, and then neglecting the double dispersion terms in the subtracted Mandelstam representation. This method closely resembles the one developed by Cini and Fubini.

a) The Kinematics of Pion-Pion Scattering

The general pion-pion scattering graph



gives rise to a T function of the general form

$$T(s, t, u) = A(s, t, u)\, \delta_{\alpha\beta}\delta_{\gamma\delta} + B(s, t, u)\, \delta_{\alpha\gamma}\, \delta_{\beta\delta} + C(s, t, u)\, \delta_{\alpha\delta}\delta_{\beta\gamma} \tag{8.1}$$

where $s = -(p_1 + p_2)^2$, $t = -(p_1 + p_3)^2$, $u = -(p_1 + p_4)^2$ and $s + t + u = 4m^2$. The relation between the functions A, B, C, and the amplitudes in particular i-spin states is given by (see appendix B)

$$\begin{aligned} A^{(0)} &= 3A + B + C \\ A^{(1)} &= B - C \\ A^{(2)} &= B + C. \end{aligned} \tag{8.2}$$

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[29] Chew and Low, Phys. Rev. **101**, 1570 (1956).
[30] Chew and Mandelstam, Phys. Rev. **119**, 467 (1960).

We obtain much information from crossing symmetry. Under the interchange $(p_2, \beta) \leftrightarrow (p_3, \gamma)$, $s \leftrightarrow t$ and also $A \leftrightarrow B$, i. e.

$$\begin{aligned} A\,(s, t, u) &= B\,(t, s, u) \\ C\,(s, t, u) &= C\,(t, s, u)\,. \end{aligned} \tag{8.3a}$$

Under the interchange $(p_2, \beta) \leftrightarrow (p_4, \delta)$, $s \leftrightarrow u$ and $A \leftrightarrow C$, so that

$$\begin{aligned} A\,(s, t, u) &= C\,(u, t, s) \\ B\,(s, t, u) &= B\,(u, t, s)\,. \end{aligned} \tag{8.3b}$$

Under the interchange $(p_2, \beta) \leftrightarrow (p_1, \alpha)$, $t \leftrightarrow u$ and $B \leftrightarrow C$, so that

$$\begin{aligned} A\,(s, t, u) &= A\,(s, u, t) \\ B\,(s, t, u) &= C\,(s, u, t)\,. \end{aligned} \tag{8.3c}$$

In the center of mass system for reaction I $(p_2 + p_1 \rightarrow (-p_3) + (-p_4))$, we have

$$\begin{aligned} s &= 4(q^2 + m^2) \\ t &= -2q^2(1 - \cos\theta) \\ u &= -2q^2(1 + \cos\theta)\,. \end{aligned} \tag{8.4}$$

Thus the interchange $t \leftrightarrow u$ corresponds to a change of sign of $\cos\theta$. Crossing symmetry shows that A and $(B + C)$ are invariant under this transformation, while $(B - C)$ changes sign. This is necessary because of the boson character of the pions: even i-spin amplitudes should be even in $\cos\theta$, and vice versa. If we write

$$A^{(i)}(q, \cos\theta) = \sum_l (2l + 1)\, A_l^{(i)}\,(q)\, P_l\,(\cos\theta) \tag{8.5}$$

then we can write

$$A_l^{(i)}(q) = \frac{\sqrt{q^2 + m^2}}{q}\, e^{i\delta_l^{(i)}(q)} \sin\delta_l^{(i)}(q) \tag{8.6}$$

with the phase shift $\delta_l^{(i)}(q)$ real for $q^2 \leq 3m^2$. As mentioned before, the reason for this large range is the G-invariance, because of which two pions must go into at least four in an inelastic process. We may write (8.6) as follows.

$$\operatorname{Im} \frac{1}{A_l^{(i)}(q)} = -\frac{q}{\sqrt{q^2 + m^2}} \quad 0 \leq q^2 \leq 3m^2. \tag{8.7}$$

The only other point to be mentioned in this "kinematical" section is that the Mandelstam representation for the pion-pion scattering amplitude leads to analyticity of the individual partial wave amplitudes in the q^2 plane, with cuts on the real axis from 0 to ∞ and from $-m^2$ to $-\infty$. This can be seen directly, or by setting $M = m$ in section VIIe and dropping the pole terms that occur there.

b) The Effective Range Approach

We make use of the above analyticity properties to write the following representation for

$$A_l^{(i)}(\nu) = \frac{1}{\pi}\int_0^{\infty}\frac{d\nu'}{\nu'-\nu}\,\mathrm{Im}\,A_l^{(i)}(\nu') + \frac{1}{\pi}\int_{-\infty}^{-1}\frac{d\nu'}{\nu'-\nu}\,\mathrm{Im}\,A_l^{(i)}(\nu') \tag{8.8}$$

where we used the notation $\nu = q^2$ and set the pion mass equal to unity. This form was written down without subtractions. In our discussion of the general problem of subtractions, we noted that for pion-pion scattering there may be need for an overall subtraction constant. Even if such a constant were not necessary, we might find it useful to make one subtraction to simulate, to some extent, the contributions from the inelastic processes which we shall neglect. We make the subtraction in the S-wave amplitude; this is consistent with the idea that the constant should somehow represent high energy effects, i. e. the short range effects not felt by the centrifugally excluded higher l waves. We therefore write

$$A_0^{(i)}(\nu) = a_i + \frac{\nu-\nu_0}{\pi}\int_0^{\infty} d\nu'\,\frac{\mathrm{Im}\,A_0^{(i)}(\nu')}{(\nu'-\nu)\,(\nu'-\nu_0)} + \frac{\nu-\nu_0}{\pi}\int_{-\infty}^{-1} d\nu'\,\frac{\mathrm{Im}\,A_0^{(i)}(\nu')}{(\nu'-\nu)\,(\nu'-\nu_0)}\,. \tag{8.9}$$

For higher l values, we recall that at the origin[31] $A_l^{(i)}(\nu) \sim c\nu^l$ so that it is permissible to write

$$\frac{A_l^{(i)}(\nu)}{\nu^l} = \frac{1}{\pi}\int_0^{\infty} d\nu'\,\frac{\mathrm{Im}\,A_l^{(i)}(\nu')}{\nu'^{\,l}\,(\nu'-\nu)} + \frac{1}{\pi}\int_{-\infty}^{-1} d\nu'\,\frac{\mathrm{Im}\,A_l^{(i)}(\nu')}{\nu'^{\,l}(\nu'-\nu)}\,. \tag{8.10}$$

We will only consider S and P waves, so that we specialize this to $l = 1$ $(i = 1)$:

$$A_1^{(1)}(\nu) = \frac{\nu}{\pi}\int_0^{\infty}\frac{d\nu'}{\nu'}\,\frac{\mathrm{Im}\,A_1^{(1)}(\nu')}{\nu'-\nu} + \frac{\nu}{\pi}\int_{-\infty}^{-1}\frac{d\nu'}{\nu'}\,\frac{\mathrm{Im}\,A_1^{(1)}(\nu')}{\nu'-\nu}\,. \tag{8.11}$$

For the subtraction constant it is convenient to pick the amplitude at the symmetric point

$$s = t = u = \frac{4}{3}$$

where we have, by crossing symmetry,

$$A = B = C = -\frac{1}{3}\,\lambda$$

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[31] This is generally true when the interaction has a finite range. Here it is seen to follow from the properties of the Q_l functions of sec. VIIe.

i. e. $\nu_0 = -{}^2/_3$ and

$$A_0^{(i)}\left(-\frac{2}{3}\right) = -\frac{1}{3}\begin{pmatrix}5\\0\\2\end{pmatrix}\lambda \qquad \begin{matrix} i=2\\ i=1\\ i=0.\end{matrix} \tag{8.12}$$

If we knew

$$\operatorname{Im} A_l^{(i)}(\nu) = f_l^{(i)}(\nu)$$

for $\nu < -1$, these relations, together with the unitarity condition (8.7) would yield integral equations for the amplitude. To discuss the structure of these equations, we write the solution $A_l^{(i)}(\nu)$ in the form

$$A_l^{(i)}(\nu) = \frac{N_l^{(i)}(\nu)}{D_l^{(i)}(\nu)} \tag{8.14}$$

where $N_l^{(i)}(\nu)$ and $D_l^{(i)}(\nu)$ are defined to be real for $\nu \geq 0$ and $\nu \leq -1$ respectively, i. e. $D_l^{(i)}(\nu)$ is to be analytic in the cut ν-plane with singularities on the positive real axis only, and $N_l^{(i)}(\nu)$ is to be analytic in the cut plane with singularities on the cut from -1 to $-\infty$. We then have

$$D_l^{(i)}(\nu) = 1 + \frac{\nu + \frac{2}{3}}{\pi}\int_0^\infty d\nu' \frac{\operatorname{Im} D_l^{(i)}(\nu')}{(\nu'-\nu)\left(\nu'+\frac{2}{3}\right)} \tag{8.15}$$

where we have normalized $D_l^{(i)}(\nu)$ to unity at the point of maximum symmetry. However

$$\operatorname{Im} D_l^{(i)}(\nu) = N_l^{(i)}(\nu) \operatorname{Im} \frac{1}{A_l^{(i)}(\nu)} = -\sqrt{\frac{\nu}{\nu+1}}\, N_l^{(i)}(\nu) \tag{8.16}$$

so that we get

$$D_l^{(i)}(\nu) = 1 - \frac{\nu + \frac{2}{3}}{\pi}\int_0^\infty d\nu' \sqrt{\frac{\nu'}{\nu'+1}}\, \frac{N_l^{(i)}(\nu')}{(\nu'-\nu)\left(\nu'+\frac{2}{3}\right)}. \tag{8.17}$$

The other equation is, for $l = 0$

$$N_0^{(i)}(\nu) = a_i + \frac{\nu + \frac{2}{3}}{\pi}\int_{-\infty}^{-1} \frac{d\nu'}{\nu'+\frac{2}{3}}\, \frac{\operatorname{Im} \mathrm{N}_0^{(i)}(\nu')}{\nu'-\nu}. \tag{8.18}$$

But

$$\operatorname{Im} \mathrm{N}_0^{(i)}(\nu) = D_0^{(i)}(\nu) f_0^{(i)}(\nu) \qquad \nu \leq -1. \tag{8.19}$$

Hence

$$N_0^{(i)}(\nu) = a_i + \frac{\nu + \frac{2}{3}}{\pi} \int_{-\infty}^{-1} d\nu' \, \frac{D_0^{(i)}(\nu')\, f_0^{(i)}(\nu')}{(\nu' - \nu)\left(\nu' + \frac{2}{3}\right)} \tag{8.20}$$

$$N_1^{(1)}(\nu) = \frac{\nu}{\pi} \int_{-\infty}^{-1} d\nu' \, \frac{D_1^{(1)}(\nu')\, f_1^{(1)}(\nu')}{\nu'(\nu' - \nu)}. \tag{8.21}$$

The effective range theory of low energy nucleon-nucleon scattering is obtained when the left-hand cut is replaced by a pole, whose position and residue are adjustable constants. In this problem, crossing symmetry allows us to calculate the discontinuity across the left-hand cut in a certain approximation. In this way no new constants are introduced into the theory, but the price to be paid for this is that a coupled set of nonlinear integral equations is obtained, which can, of course be solved on an electronic computing machine, but which are quite untransparent.
The expression

$$f_l^{(i)}(\nu) = \frac{1}{\nu} \int_0^{-\nu-1} d\nu'' \, P_l\left(1 + 2\,\frac{\nu'' + 1}{\nu}\right) \sum_{j=0}^{2} \alpha_{ij}\, A_{\mathrm{I}}^{(j)}\left(\nu'', 1 + 2\,\frac{\nu + 1}{\nu''}\right) \tag{8.22}$$

where

$$\alpha_{ij} = \frac{1}{3}\begin{pmatrix} 2 & 6 & 10 \\ 2 & 3 & -5 \\ 2 & -3 & 1 \end{pmatrix} \tag{8.23}$$

and

$$A_1^{(i)}(\nu'', x) = \sqrt{\frac{\nu'' + 1}{\nu''}} \sum_l (2l + 1) \sin^2 \delta_l^{(i)}(\nu'')\, P_l(x) \tag{8.24}$$

can be obtained straightforwardly, though tediously, from the Mandelstam representation, if one makes use of the fact that in the range of variables of interest to us ($s \leq 16$, $t \leq 16$ simultaneously)

$$A_{23}\,(-2\nu(1 \pm \cos\theta);\ -2\nu(1 \mp \cos\theta)) = 0$$

where A_{23} is a typical weight function in the representation.
In the expression for $A_1^{(i)}$ only S and P wave terms are kept. This gives rise to a set of coupled equations which have not yet been studied exhaustively. Chew and Mandelstam point out that the fact that there is no two-pion bound state limits the values of the parameter λ. For those values of λ for which a zero appears in $D_l^{(i)}(\nu)$ in the range $-1 < \nu \leq 0$ the scattering amplitude would have an unwanted pole. From preliminary studies of the equations, Chew and Mandelstam[31a]) conclude that there is no resonance in the S state, but that there might be

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[31a]) Chew and Mandelstam, Phys. Rev. **119**, 467 (1960); Chew, Mandelstam and Noyes, Phys. Rev. **119**, 478 1960).

one in the P state. Such a resonance would be of great significance in the study of the electromagnetic form factor of the nucleon[32]).

At this point it is necessary to say a few words about the problem of uniqueness. It is clear that aside from any nonuniqueness which might appear when the equations are finally solved, there is an inherent nonuniqueness known as the *CDD* ambiguity[33]). In this problem it manifests itself in the possibility of adding to $D_l^{(i)}(\nu)$ a finite sum of terms of the form $\alpha_\varkappa/(\nu - \nu_\varkappa)$ $(\nu_\varkappa > 0)$. This would not affect the analyticity properties, but would change the scattering amplitude. Studies of the *CDD* ambiguity in simple models indicate that its appearance in the solutions of dispersion equations is associated with an ambiguity present in the canonical formulation of field theory[34]). There the possibility exists of including additional particles in the theory and adjusting the parameters so that their renormalized masses lie in the continuum, i. e. that they be unstable. Such particles will not give rise to "one-particle" poles in the dispersion relations, but they will certainly produce resonances in the scattering amplitude at energies in the neighborhood of their masses. In the equations above the assumption was made that such particles are not present. If however such unstable particles are expected in the theory on other grounds[35]), then such terms could very well be present, though the above method of introducing them is presumably a little too primitive. An experimental finding of resonances could have its origin in the low energy dynamics *or* in the unstable particles, and therefore the *CDD* ambiguity cannot be trivially dismissed.

c) The Single Dispersion Integral Method

This method consists of making a certain number of subtractions, and treating the single dispersion terms only[36]). The effect of the double dispersion terms (see sec. VIId) and of the inelastic contributions is either neglected or represented by one or more arbitrary constants. One does not know a priori how many subtractions should be made, i. e. in how many partial waves should the absorptive part be treated carefully. The consistency of whatever approximation one makes can be checked, as will be seen later. Consider pion-pion scattering again. We shall keep only the S and P absorptive parts, and then write terms for the three channels:

$$\begin{aligned} A(s,t,u) = {} & \frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s}\{A_{\mathrm{I}}^{(0)}(s') + 3\cos\theta_{\mathrm{I}}(s',t,u)A_{\mathrm{I}}^{(1)}(s')\} + \\ & + \frac{1}{\pi}\int_4^\infty \frac{du'}{u'-u}\{A_{\mathrm{II}}^{(0)}(u') + 3\cos\theta_{\mathrm{II}}(s,t,u')\,A_{\mathrm{II}}^{(1)}(u')\} + \\ & + \frac{1}{\pi}\int_4^\infty \frac{dt'}{t'-t}\{A_{\mathrm{III}}^{(0)}(t') + 3\cos\theta_{\mathrm{III}}(s,t',u)\,A_{\mathrm{III}}^{(1)}(t')\} \end{aligned} \tag{8.25}$$

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[32]) Frazer and Fulco, Phys. Rev. Letters 2, 365 (1959).

[33]) Castillejo, Dalitz and Dyson, Phys. Rev. 101, 543 (1956).

[34]) See Norton and Klein, Phys. Rev. 109, 991 (1958) and references to earlier work.

[35]) See for example J. J. Sakurai, Annals of Physics (to be published).

[36]) This method is briefly discussed in reference[30]).

with similar equations for B and C. For process I, we have $s = 4(\nu + 1)$; $t = -2\nu(1 - \cos\theta_{\mathrm{I}})$; $u = -2\nu(1 + \cos\theta_{\mathrm{I}})$ so that

$$\cos\theta_{\mathrm{I}}(s', t, u) = \frac{t - u}{s' - 4}. \tag{8.26a}$$

For process II, we find that

$$\cos\theta_{\mathrm{II}}(s, t, u') = \frac{t - s}{u' - 4} \tag{8.26b}$$

and for process III

$$\cos\theta_{\mathrm{III}}(s, t', u) = \frac{s - u}{t' - 4}. \tag{8.26c}$$

Thus we have with obvious abbreviations

$$\begin{aligned} A &= I_A^{(0)}(s) + (t-u)\, I_A^{(1)}(s) + J_A^{(0)}(u) + (t-s)\, J_A^{(1)}(u) + K_A^{(0)}(t) + (s-u) K_A^{(1)}(t) \\ B &= I_B^{(0)}(s) + \cdots \\ C &= I_C^{(0)}(s) + \cdots \end{aligned} \tag{8.27}$$

The eighteen integrals are not independent because of crossing symmetry. It is easy to check that only three integrals remain because

$$\begin{aligned} & I_A^{(1)} = J_C^{(1)} = K_B^{(1)} = 0 \\ & I_A^{(0)} = J_C^{(0)} = K_B^{(0)} \equiv L \\ & I_B^{(0)} = I_C^{(0)} = J_A^{(0)} = J_B^{(0)} = K_A^{(0)} = K_C^{(0)} \equiv M \\ & I_B^{(1)} = -I_C^{(1)} = -J_A^{(1)} = J_B^{(1)} = K_A^{(1)} = -K_C^{(1)} \equiv N. \end{aligned} \tag{8.28}$$

Thus

$$\begin{aligned} A(s,t,u) &= L(s) + M(u) - (t-s)\, N(u) + M(t) + (s-u)\, N(t) \\ B(s,t,u) &= M(s) + (t-u)\, N(s) + M(u) + (t-s)\, N(u) + L(t) \\ C(s,t,u) &= M(s) - (t-u)\, N(s) + L(u) + M(t) - (s-u)\, N(t) \end{aligned} \tag{8.29}$$

where

$$\begin{aligned} L(s) &= \frac{1}{\pi}\int_4^\infty d\alpha\, \frac{l(\alpha)}{\alpha - s} \\ M(s) &= \frac{1}{\pi}\int_4^\infty d\alpha\, \frac{m(\alpha)}{\alpha - s} \\ N(s) &= \frac{1}{\pi}\int_4^\infty d\alpha\, \frac{n(\alpha)}{\alpha - s}. \end{aligned} \tag{8.30}$$

We have thus taken into account the analyticity properties and crossing symmetry. It remains to take into account the unitarity condition, in a certain approximation. We recombine the amplitudes into *i*-spin amplitudes:

$$
\begin{aligned}
A^{(0)} &= 3L(s) + 2M(s) + 4M(u) + L(u) + 4M(t) + L(t) + \\
&\qquad + 2\,(s-u)\,N(t) - 2\,(t-s)\,N(u) \\
A^{(1)} &= M(u) - L(u) - M(t) + L(t) + 2\,(t-u)\,N(s) + \\
&\qquad + (t-s)\,N(u) + (s-u)\,N(t) \\
A^{(2)} &= 2M(s) + M(u) + L(u) + M(t) + L(t) + (t-s)\,N(u) - (s-u)\,N(t).
\end{aligned}
\tag{8.31}
$$

For values of *s*, *t*, *u* lying in the physical region for process I, say, only the integrals involving *s* have an imaginary part. Thus

$$
\begin{aligned}
\operatorname{Im} A^{(0)} &= 3l(s) + 2m(s) \\
\operatorname{Im} A^{(1)} &= 2\,\frac{t-u}{s-4}\,n(s) = 2\cos\theta_{\mathrm{I}}\,n(s) \\
\operatorname{Im} A^{(2)} &= 2m(s).
\end{aligned}
\tag{8.32}
$$

Further, if we write

$$
\begin{aligned}
\operatorname{Im} A^{(0)} &\approx \sqrt{\frac{\nu+1}{\nu}}\,\sin^2\delta_0^{(0)}(\nu) \\
\operatorname{Im} A^{(1)} &\approx 3\sqrt{\frac{\nu+1}{\nu}}\,\cos\theta\,\sin^2\delta_1^{(1)}(\nu) \\
\operatorname{Im} A^{(2)} &\approx \sqrt{\frac{\nu+1}{\nu}}\,\sin^2\delta_0^{(2)}(\nu)
\end{aligned}
\tag{8.33}
$$

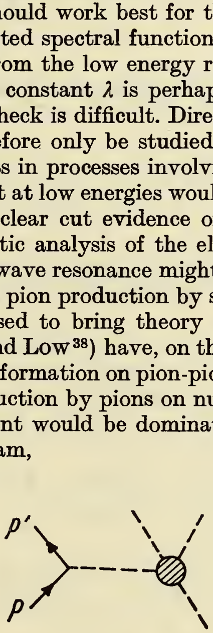
we find expressions for the weight functions $l(\alpha)$, $m(\alpha)$, $n(\alpha)$ in terms of the amplitudes. Projecting partial waves out in eq. (8.31), we obtain

$$
\begin{aligned}
A_l^{(0)}(\nu) &= \sqrt{\frac{\nu+1}{\nu}}\,e^{i\delta_l^{(0)}}\sin\delta_l^{(0)} = \\
&= \frac{1}{2}\int_{-1}^{1} dx\,P_l(x)\,A^{(0)}(\nu,x) = \\
&= \frac{1}{2}\int_{-1}^{1} dx\,P_l(x)\left\{\frac{1}{\pi}\int_4^\infty d\alpha\,\frac{3l(\alpha)+2m(\alpha)}{\alpha-s} + \right. \\
&\quad + \frac{1}{\pi}\int_4^\infty d\alpha\,(4m(\alpha)+l(\alpha))\left(\frac{1}{\alpha+2\nu\,(1-x)} + \frac{1}{\alpha+2\nu\,(1+x)}\right) +
\end{aligned}
$$

$$
\left. \begin{aligned}
&+ 2\,(6\nu + 4 + 2\nu x)\,\frac{1}{\pi}\int_4^\infty d\alpha\,\frac{n(\alpha)}{\alpha + 2\nu\,(1 - x)} + \\
&+ 2\,(6\nu + 4 - 2\nu x)\,\frac{1}{\pi}\int_4^\infty d\alpha\,\frac{n(\alpha)}{\alpha + 2\nu\,(1 + x)}
\end{aligned} \right\} \tag{8.34}
$$

and similar equations for the other *i*-spin amplitudes[37]. If we keep only $A_0^{(0)}$, $A_1^{(1)}$ and $A_0^{(2)}$ we get a set of coupled equations. Re-insertion of the solutions of these equations into (8.34) for $l > 0$ allows us to compute these *real* amplitudes. If they turn out to be small, then the assumption that they are real is justifiable. If they turn out to be large, then presumably more partial waves have to be treated exactly in the elastic approximation.

The approximation method should work best for the pion-pion problem as there the region in which the neglected spectral functions and absorptive parts for the inelastic processes lie, is far from the low energy region, and the replacement of these terms by an adjustable constant λ is perhaps a good approximation. Unfortunately an experimental check is difficult. Direct experiments are impossible, and the interaction can therefore only be studied indirectly, primarily through "final state interaction" effects in processes involving more than one pion in the final state. Any dramatic effect at low energies would have manifested itself in the τ decay distribution, but no clear cut evidence of a low energy resonance has appeared. A dispersion-theoretic analysis of the electromagnetic form factors of the nucleon indicate that a P-wave resonance might be present at higher energies, and in the analysis of multiple pion production by statistical models indicate that such a resonance could be used to bring theory in better agreement with the experimental results. Chew and Low[38] have, on the basis of perturbation theory, made a suggestion as to how information on pion-pion scattering could be obtained from an analysis of pion production by pions on nucleons. They observe that the pion production matrix element would be dominated by the pole at $-(p-p')^2 = m^2$, coming from the diagram,



in the vicinity of that pole. The residue of the pole is the product of the meson-nucleon coupling constant and the meson-meson scattering amplitude with all mesons on the mass shell. Experimentally we can of course never reach the pole, but a strong resonance could make itself felt in the low-nucleon-recoil region.

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[37] Here no subtraction has been made in the S-wave absorptive part but this is easily remedied.

[38] Chew and Low, Phys. Rev. **113**, 1640 (1959).

IX. Meson-Nucleon Scattering

In this section we give a brief discussion of pion-nucleon scattering using the single dispersion integral method. With a view to studying the possible influence of the second resonance ($i = {}^1/_2$, $j = {}^3/_2$, $l = 2$) on low energy scattering, we write the single dispersion integrals for S, P and D waves in channels I and II, corresponding to scattering ("crossed" and "uncrossed"), and for the time being we leave the number of subtractions in channel III (which corresponds to $N + \overline{N} \rightarrow \pi + \pi$), open. We thus write

$$\begin{aligned}
A^{(\pm)}(s, t, u) = {} & \frac{1}{\pi} \int\limits_{(M+m)^2}^{\infty} \frac{ds'}{s' - s} \left\{ \alpha_0^{\pm}(s') + 3 \cos \theta_{\mathrm{I}}(s', t)\, \alpha_1^{\pm}(s') + \right. \\
& \left. + 5 \, \frac{3 \cos^2 \theta_{\mathrm{I}}(s', t) - 1}{2} \, \alpha_2^{\pm}(s') \right\} + \\
& + \frac{1}{\pi} \int\limits_{(M+m)^2}^{\infty} \frac{du'}{u' - u} \left\{ \tilde{\alpha}_0^{\pm}(u') + 3 \cos \theta_{\mathrm{II}}(u', t)\, \tilde{\alpha}_1^{\pm}(u') + \right. \\
& \left. + 5 \, \frac{3 \cos^2 \theta_{\mathrm{II}}(u', t) - 1}{2} \, \tilde{\alpha}_2^{\pm}(u') \right\} + \\
& + \frac{1}{\pi} \int\limits_{4m^2}^{\infty} \frac{dt'}{t' - t} \sum_{l=0}^{L} (2l + 1)\, P_l(\cos \theta_{\mathrm{III}}(s, t', u))\, g_l^{\pm}(t')
\end{aligned} \tag{9.1}$$

and

$$\begin{aligned}
B^{\pm}(s, t, u) = {} & \frac{g^2}{M^2 - s} \mp \frac{g^2}{M^2 - u} + \\
& + \frac{1}{\pi} \int\limits_{(M+m)^2}^{\infty} \frac{ds'}{s' - s} \left\{ \beta_0^{\pm}(s') + 3 \cos \theta_{\mathrm{I}}(s', t)\, \beta_1^{\pm}(s') + \right. \\
& \left. + 5 \, \frac{3 \cos^2 \theta_{\mathrm{I}}(s', t) - 1}{2} \, \beta_2^{\pm}(s') \right\} + \\
& + \frac{1}{\pi} \int\limits_{(M+m)^2}^{\infty} \frac{du'}{u' - u} \left\{ \tilde{\beta}_0^{\pm}(u') + 3 \cos \theta_{\mathrm{II}}(u', t)\, \tilde{\beta}_1^{\pm}(u') + \right. \\
& \left. + 5 \, \frac{3 \cos^2 \theta_{\mathrm{II}}(u', t) - 1}{2} \, \tilde{\beta}_2^{\pm}(u') \right\} + \\
& + \frac{1}{\pi} \int\limits_{4m^2}^{\infty} \frac{dt'}{t' - t} \sum_{l=0}^{L} (2l + 1)\, P_l(\cos \theta_{\mathrm{III}}(s, t', u))\, h_l^{\pm}(t').
\end{aligned} \tag{9.2}$$

In the center of mass system for channel I, $s = W^2$, and $\cos\theta_{\mathrm{I}} = 1 + t/(2q^2(s))$. For channel II, $u = W^2$ and $\cos\theta_{\mathrm{II}} = 1 + t/(2q^2(u))$ and for channel III we have $t = 4(q^2 + m^2) = 4(p^2 + M^2) = 4\omega^2 = 4E^2$; $s = -p^2 - q^2 - 2pq\cos\theta_{\mathrm{III}}$ and $u = -p^2 - q^2 + 2pq\cos\theta_{\mathrm{III}}$ so that

$$\cos\theta_{\mathrm{III}} = \frac{u - s}{4pq}. \tag{9.3}$$

The crossing symmetry conditions (7.3) yield

$$\begin{aligned} \tilde{\alpha}_i^{\pm}(x) &= \pm\,\alpha_i^{\pm}(x) \\ \tilde{\beta}_i^{\pm}(x) &= \mp\,\beta_i^{\pm}(x) \\ g_{\text{even}}^{-}(x) &= h_{\text{even}}^{+}(x) = 0 \\ g_{\text{odd}}^{+}(x) &= h_{\text{odd}}^{-}(x) = 0. \end{aligned} \tag{9.4}$$

The next step is to determine the weight functions by using the unitarity condition. In the region in which reaction I is "physical", only the s integrals have imaginary parts. Thus

$$\operatorname{Im} A^{(\pm)}(s, t, u) = \alpha_0^{\pm}(s) + 3\cos\theta_{\mathrm{I}}\,\alpha_1^{\pm}(s) + 5\,\frac{3\cos^2\theta_{\mathrm{I}} - 1}{2}\,\alpha_2^{\pm}(s)$$

$$\operatorname{Im} B^{\pm}(s, t, u) = \beta_0^{\pm}(s) + 3\cos\theta_{\mathrm{I}}\,\beta_1^{\pm}(s) + 5\,\frac{3\cos^2\theta_{\mathrm{I}} - 1}{2}\,\beta_2^{\pm}(s). \tag{9.5}$$

Using the formulae (A.25) and keeping only the a_{33} and d_{13} terms, because of their relation to observed resonances, we get

$$\begin{aligned} \alpha_0^{(+)} &\approx 4\pi\,\frac{W + M}{E + M}\left(-\frac{1}{3}\operatorname{Im} d_{13}\right) - 4\pi\,\frac{W - M}{E - M}\left(-\frac{2}{3}\operatorname{Im} a_{33}\right) \\ \alpha_0^{(-)} &\approx 4\pi\,\frac{W + M}{E + M}\left(-\frac{1}{3}\operatorname{Im} d_{13}\right) - 4\pi\,\frac{W - M}{E - M}\left(+\frac{1}{3}\operatorname{Im} a_{33}\right) \\ \alpha_1^{(+)} &\approx 4\pi\,\frac{W + M}{E + M}\left(\frac{2}{3}\operatorname{Im} a_{33}\right) - 4\pi\,\frac{W - M}{E - M}\left(\frac{1}{3}\operatorname{Im} d_{13}\right) \\ \alpha_1^{(-)} &\approx 4\pi\,\frac{W + M}{E + M}\left(-\frac{1}{3}\operatorname{Im} a_{33}\right) - 4\pi\,\frac{W - M}{E - M}\left(\frac{1}{3}\operatorname{Im} d_{13}\right) \end{aligned} \tag{9.6}$$

and

$$\alpha_2^{(+)} = \alpha_2^{(-)} \approx 0.$$

The corresponding quantities $\beta_0^{(+)}$, $\beta_0^{(-)}$, $\beta_1^{(+)}$, $\beta_1^{(-)}$, are obtained from the above formulae by replacing $(W + M)/(E + M)$ by $1/(E + M)$ and $(W - M)/(E - M)$ by $-1/(E - M)$. Also

$$\beta_2^{(+)} = \beta_2^{(-)} \approx 0.$$

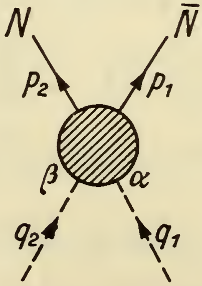
If we neglect all meson-meson interaction effects, the lower integration limits on the t' integrals are $4M^2$, so that these integrals are effectively constants. If we only keep the a_{33} terms in the expressions for the weight functions, we get a system of equations which express the partial wave amplitudes in terms of the dominant amplitude. In the limit that only first order terms in $1/M$ are kept, we get the equations of CHEW, GOLDBERGER, LOW and NAMBU[39]). If we wish to include the effects of a meson-meson interaction, we must determine the weight functions $g_l^{\pm}(t')$ and $h_l^{\pm}(t')$. To do this, we consider values of the momenta such that the reaction III is "physical". Then only the t-integrals have imaginary parts, and in particular we have

$$\begin{aligned}(\operatorname{Im} A^{\pm}(s, t, u))_{\mathrm{III}} &= \sum_{l=0}^{L} (2l+1)\, P_l(\cos\theta_{\mathrm{III}})\, g_l^{\pm}(t)\\ (\operatorname{Im} B^{\pm}(s, t, u))_{\mathrm{III}} &= \sum_{l=0}^{L} (2l+1)\, P_l(\cos\theta_{\mathrm{III}})\, h_l^{\pm}(t)\end{aligned} \tag{9.7}$$

$$\begin{aligned}g_l^{\pm}(t) &= \frac{1}{2}\int_{-1}^{1} dx\, P_l(x)\, \operatorname{Im} A^{\pm}(t, x)\\ h_l^{\pm}(t) &= \frac{1}{2}\int_{-1}^{1} dx\, P_l(x)\, \operatorname{Im} B^{\pm}(t, x).\end{aligned} \tag{9.8}$$

FRAZER and FULCO[40]) have set up an "effective range approach" analogous to the CHEW-MANDELSTAM treatment of pion-pion scattering, in terms of amplitudes for the process $\pi + \pi \to \mathrm{N} + \overline{\mathrm{N}}$ for a given angular momentum and given helicity states of the fermions. These amplitudes have simple analyticity properties and partial wave dispersion relations can thus more easily be written down. As we only want expressions for the imaginary parts of the amplitudes in a particular region, viz. the one in which the process $\pi + \pi \to \mathrm{N} + \overline{\mathrm{N}}$ is physical[41]), we do not have to choose amplitudes of a particularly simple analytic form, but can proceed straightforwardly as follows.

Consider the process described by the graph



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[39]) CHEW, GOLDBERGER, LOW and NAMBU, Phys. Rev. **106**, 1337 (1957).

[40]) FRAZER and FULCO, Phys. Rev. **117**, 1603 (1960).

[41]) Actually the amplitudes only become physical for $t \geq 4M^2$; we therefore deal with an analytic continuation to the domain $4m^2 \leq t$.

The S-matrix element for this process has the form

$$\langle p_2, p_1^{c\ \mathrm{out}} \mid q_1, q_2^{\mathrm{in}} \rangle = - \frac{(2\pi)^4\, i\, \delta(p_1 + p_2 - q_1 - q_2)}{n_{p_1}\, n_{p_2}\, n_{q_1}\, n_{q_2}\, V^2}\, T\,(p_1^c, p_2; q_1, q_2)$$

and using the reduction formulae we can obtain the following forms

$$T(p_1^c p_2; q_1 q_2) = - n_{p_2}\, n_{q_1}\, n_{q_2}\, V^{3/2} \langle p_2 \mid \bar{\mathfrak{D}}(0) \mid q_1\, q_2 \rangle\, v(p_1)$$

and

$$T(p_1^c\, p_2;\ q_1\ q_2) =$$

$$= - i n_{p_2}\, n_{q_2}\, V \int dz\, e^{-i \frac{p_1 + q_1}{2} z}\, \theta(z) \left\langle p_2 \middle| \left[\bar{\mathfrak{D}} \left(\frac{z}{2}\right), j_\alpha \left(-\frac{z}{2}\right)\right] \middle| q_2 \right\rangle v(p_1).$$

Writing

$$T(p_1^c, p_2; q_1, q_2) = \bar{u}(p_2) \left[- A_{\beta\alpha} + i\gamma\, \frac{q_1 - q_2}{2}\, B_{\beta\alpha} \right] v(p_1) \tag{9.9}$$

we find that in the approximation in which only the two-pion intermediate states are considered in the unitarity condition,

$$\bar{u}(p_2) \left[- \mathrm{Im}\, A_{\beta\alpha} + i\gamma \frac{q_1 - q_2}{2}\, \mathrm{Im}\, B_{\beta\alpha} \right] v(p_1) =$$

$$= - \frac{(2\pi)^4}{2} \int \frac{d^3 q_3}{(2\pi)^3} \frac{d^3 q_4}{(2\pi)^3} \frac{1}{4\omega_3 \omega_4}\, \delta\,(q_1 + q_2 - q_3 - q_4)$$

$$\sum_{\gamma\delta} \left(- A_{\gamma\delta} + i\gamma\, \frac{q_3 - q_4}{2}\, B_{\gamma\delta} \right) T_{\gamma\delta,\ \alpha\beta}\ (q_3 q_4;\ q_1 q_2) \tag{9.10}$$

where $T_{\gamma\delta,\ \alpha\beta}$ is the amplitude for meson-meson scattering.
In the center of mass system

$$\left.\begin{array}{lll} p_1 = (\boldsymbol{p}, E); & q_1 = (\boldsymbol{q}, \omega); & q_3 = (\boldsymbol{q}', \omega) \\ p_2 = (-\boldsymbol{p}, E); & q_2 = (-\boldsymbol{q}, \omega); & q_4 = (-\boldsymbol{q}', \omega) \end{array}\right\} ;\ \omega = E \tag{9.11}$$

we have

$$\bar{u}(p_2)\, v(p_1) = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{M} \tag{9.12}$$

and

$$\bar{u}(p_2)\, i\gamma Q v(p_1) = \frac{E}{M} \boldsymbol{\sigma} \cdot \boldsymbol{Q} - \frac{\boldsymbol{Q} \cdot \boldsymbol{p}}{M(E + M)}\, \boldsymbol{\sigma} \cdot \boldsymbol{p} \tag{9.13}$$

with $\boldsymbol{Q} = \boldsymbol{q}_1$ or $\boldsymbol{q}_3$. Thus eq. (9.10) becomes

$$
\begin{aligned}
-\frac{1}{M}\left(\operatorname{Im} A_{\beta\alpha} + \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{E+M}\operatorname{Im} B_{\beta\alpha}\right)\boldsymbol{\sigma}\cdot\boldsymbol{p} + \frac{E}{M}\operatorname{Im} B_{\beta\alpha}\boldsymbol{\sigma}\cdot\boldsymbol{q} = \\
= -\frac{1}{8\pi^2}\int\frac{q'\,E'\,d\,E'}{4E'^2}\,\delta(2E-2E')\int d\,\Omega_{\hat{q}'}\sum_{\gamma\delta}\left\{-\frac{1}{M}\left(A_{\gamma\delta}+\frac{\boldsymbol{q}'\cdot\boldsymbol{p}}{E+M}B_{\gamma\delta}\right)\boldsymbol{\sigma}\cdot\boldsymbol{p} + \right. \\
\left. + \frac{E}{M}B_{\gamma\delta}\boldsymbol{\sigma}\cdot\boldsymbol{q}'\right\}T_{\gamma\delta,\ \alpha\beta} = \qquad (9.14) \\
= -\frac{1}{64\pi^2}\frac{q}{E}\int d\,\Omega_{\hat{q}'}\sum_{\gamma\delta}\left\{-\frac{1}{M}\left(A_{\gamma\delta}+\frac{\boldsymbol{q}'\cdot\boldsymbol{p}}{E+M}B_{\gamma\delta}\right)\boldsymbol{\sigma}\cdot\boldsymbol{p} + \frac{E}{M}B_{\gamma\delta}\,\boldsymbol{\sigma}\cdot\boldsymbol{q}'\right\}T_{\gamma\delta\ \alpha\beta}
\end{aligned}
$$

and using the forms

$$A_{\beta\alpha} = A^{(+)}\,\delta_{\alpha\beta} + \frac{1}{2}\,[\tau_\beta, \tau_\alpha]\,A^{(-)} \quad \text{etc.}$$

and

$$T_{\gamma\delta,\,\alpha\beta} = \frac{1}{3}\,(A^{(0)} - A^{(2)})\,\delta_{\alpha\beta}\,\delta_{\gamma\delta} + \frac{1}{2}\,(A^{(1)} + A^{(2)})\,\delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{1}{2}\,(A^{(1)} - A^{(2)})\,\delta_{\alpha\delta}\delta_{\beta\gamma}$$

we can carry out the i-spin sum to obtain the two equations

$$
\begin{aligned}
-\frac{1}{M}\left(\operatorname{Im} A^{(+)} + \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{E+M}\operatorname{Im} B^{(+)}\right)\boldsymbol{\sigma}\cdot\boldsymbol{p} + \frac{E}{M}\operatorname{Im} B^{(+)}\,\boldsymbol{\sigma}\cdot\boldsymbol{q} = \\
= -\frac{q}{64\,\pi^2 E}\int d\,\Omega_{\hat{q}'}\left[-\frac{1}{M}\left(A^{(+)}+\frac{\boldsymbol{q}'\cdot\boldsymbol{p}}{E+M}B^{(+)}\right)\boldsymbol{\sigma}\cdot\boldsymbol{p} + \frac{E}{M}B^{(+)}\boldsymbol{\sigma}\cdot\boldsymbol{q}'\right]A^{(0)}
\end{aligned}
\qquad (9.15)
$$

and

$$
\begin{aligned}
-\frac{1}{M}\left(\operatorname{Im} A^{(-)} + \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{E+M}\operatorname{Im} B^{(-)}\right)\boldsymbol{\sigma}\cdot\boldsymbol{p} + \frac{E}{M}\operatorname{Im} B^{(-)}\boldsymbol{\sigma}\cdot\boldsymbol{q} \\
= \frac{q}{64\,\pi^2\,E}\int d\,\Omega_{\hat{q}'}\left[-\frac{1}{M}\left(A^{(-)}+\frac{\boldsymbol{q}'\cdot\boldsymbol{p}}{E+M}B^{(-)}\right)\boldsymbol{\sigma}\cdot\boldsymbol{p} + \frac{E}{M}B^{(-)}\boldsymbol{\sigma}\cdot\boldsymbol{q}'\right]A^{(1)}
\end{aligned}
\qquad (9.16)
$$

where $A^{(0)}$ and $A^{(1)}$ are the meson-meson scattering amplitudes for $i = 0$ and 1. If we now write

$$
\begin{aligned}
A^{(\pm)} &= \sum_l\,(2l+1)\,X_l^{(\pm)}\,P_l\,(\hat{p}\cdot\hat{q}\,') \\
B^{(\pm)} &= \sum_l\,(2l+1)\,Y_l^{(\pm)}\,P\,(\hat{p}\cdot\hat{q}\,') \qquad (9.17) \\
A^{(i)} &= \sum_l\,(2l+1)\,A_l^{(i)}\,P_l(\hat{q}'\cdot\hat{q}\,) \qquad (i = 0, 1)
\end{aligned}
$$

we can obtain expressions for $g_l^{(\pm)}$ and $h_l^{(\pm)}$ (see eq. (9.8)) in terms of $X_l^{(\pm)}$, $Y_l^{(\pm)}$ and the meson-meson scattering amplitudes. To do this we use

$$\int d\Omega_{\hat{q}'}\, P_l(\hat{q}'\cdot\hat{p})\, P_m(\hat{q}'\cdot\hat{q}) = \frac{4\pi}{2l+1}\,\delta_l^m\, P_l(\hat{p}\cdot\hat{q}) \tag{9.18a}$$

$$\int d\Omega_{\hat{q}'}\, P_l(\hat{q}'\cdot\hat{p})\, \hat{q}'\cdot\hat{p}\, P_m(\hat{q}'\cdot\hat{q}) =$$

$$= \int d\Omega_{\hat{q}'}\left[\frac{l}{2l+1}\, P_{l-1}(\hat{p}\cdot\hat{q}') + \frac{l+1}{2l+1}\, P_{l+1}(\hat{p}\cdot\hat{q}')\right] P_m(\hat{q}'\cdot\hat{q}) =$$

$$= \frac{4\pi}{2l+1}\left[\frac{l}{2l-1}\,\delta_{l-1}^m\, P_{l-1}(\hat{p}\cdot\hat{q}) + \frac{l+1}{2l+3}\,\delta_{l+1}^m\, P_{l+1}(\hat{p}\cdot\hat{q})\right. \tag{9.18b}$$

and

$$\int d\Omega_{\hat{q}'}\, P_l(\hat{p}\cdot\hat{q}')\,\boldsymbol{\sigma}\cdot\hat{q}'\, P_m(\hat{q}'\cdot\hat{q}) =$$

$$= \boldsymbol{\sigma}\cdot\nabla_{\hat{p}}\int d\Omega_{\hat{q}'}\,\frac{1}{2l+1}\,[P_{l+1}(\hat{p}\cdot\hat{q}') - P_{l-1}(\hat{p}\cdot\hat{q}')]\, P_m(\hat{q}'\cdot\hat{q}) =$$

$$= \frac{4\pi}{2l+1}\,\boldsymbol{\sigma}\cdot\nabla_{\hat{p}}\left[\frac{1}{2l+3}\,\delta_{l+1}^m\, P_{l+1}(\hat{p}\cdot\hat{q}) - \frac{1}{2l-1}\,\delta_{l-1}^m\, P_{l-1}(\hat{p}\cdot\hat{q})\right] =$$

$$= \frac{4\pi}{2l+1}\,\boldsymbol{\sigma}\cdot\hat{q}\left[\frac{1}{2l+3}\,\delta_{l+1}^m\, P'_{l+1}(\hat{p}\cdot\hat{q}) - \frac{1}{2l-1}\,\delta_{l-1}^m\, P'_{l-1}(\hat{p}\cdot\hat{q})\right] \tag{9.18c}$$

and if, after carrying out the integration, we keep terms with $l = 0$ and 1 only, we obtain

$$\begin{aligned}
g_0^{(+)}(t) &= \frac{-1}{16\pi}\,\frac{q}{E}\left(X_0^{(+)}(t) + \frac{qp}{E+M}\, Y_1^{(+)}(t)\right) A_0^{(0)}(t) \\
g_1^{(+)} &= h_0^{(+)} = h_1^{(+)} \cong 0 \\
g_0^{(-)} &= h_1^{(-)} \cong 0 \\
g_1^{(-)}(t) &= \frac{1}{16\pi}\,\frac{q}{E}\, X_1^{(-)}(t)\, A_1^{(1)}(t) \\
h_0^{(-)}(t) &= \frac{1}{16\pi}\,\frac{q}{E}\, Y_0^{(-)}(t)\, A_1^{(1)}(t).
\end{aligned} \tag{9.19}$$

We assume that the meson-meson scattering amplitudes are known. To complete the set of equations, we need expressions for the quantities $X_0^{(+)}(t)$, . . . etc. This is done by projecting the S and P parts (in $\cos\theta_{\mathrm{III}}$) out of eqs. (9.1) and (9.2). For example, with $x = \cos\theta_{\mathrm{III}}$

$$X_0^{(+)}(t) = \frac{1}{2}\int_{-1}^{1} d(\cos\theta_{\mathrm{III}}(s,t,u))\,\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{dt'}{t'-t}\sum_{l=0}^{L}(2l+1)\,g_l^{+}(t')P_l(\cos\theta_{\mathrm{III}}(s,t',u)) +$$

$$+\frac{1}{2}\int_{-1}^{1} dx\left\{\frac{1}{\pi}\int_{(M+m)^2}^{\infty}\frac{ds'}{s'+p^2+q^2+2pqx}\,(\alpha_0^{(+)}(s')+\cdots) +\right.$$

$$\left.+\frac{1}{\pi}\int_{(M+m)^2}^{\infty}\frac{du'}{u'+p^2+q^2-2pqx}\,(\tilde{\alpha}_0^{(+)}(u')+\cdots)\right\}. \tag{9.20}$$

These equations are quite untransparent, and we do not therefore write them down. The formulae which appear in this section are sufficient for a calculation of the S phase shifts and the small P phase shifts in terms of the dominant pion-nucleon and pion-pion amplitudes. In conclusion, attention should be drawn to the fact that, as seen in eq. (9.19), the phases of the quantities $X_0^{(+)}(t), \ldots$ are determined by the pion-pion scattering phase shifts. This is a special example of the so-called FERMI-WATSON theorem[42].

The approximation methods reviewed in the above paper must be studied in more detail, and applied to a variety of problems in the field of low energy elementary particle reactions, before their validity can be properly estimated. It is to make the foundations of these approximation methods more accessible to interested workers, and to clear up some of the complications connected with the kinematics, spin and *i*-spin that this article has been written.

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Appendix A

Partial Wave Analysis for Meson-Nucleon Scattering

The form of the T function is

$$T = \overline{u}_\alpha^{(r)}(p')\left[-A + i\gamma\,\frac{q+q'}{2}\,B\right]_{\alpha\beta} u_\beta^{(s)}(p). \tag{A.1}$$

Using the explicit representation for the spinors $u_\beta^{(s)}(p)$:

$$u_\beta^{(s)}(p) = \frac{(M - i\gamma p)_{\beta s}}{\sqrt{2M(M+E)}} = \frac{(M - i\varrho_2\boldsymbol{\sigma}\cdot\boldsymbol{p} + \varrho_3 E)_{\beta s}}{\sqrt{2M(M+E)}} \tag{A.2}$$

and

$$\overline{u}_\alpha^{(r)}(p') = \frac{(M - i\varrho_2\boldsymbol{\sigma}\cdot\boldsymbol{p}' + \varrho_3 E')_{r\alpha}}{\sqrt{2M(M+E')}} \tag{A.3}$$

---

[42] See FUBINI, NAMBU and WATAGHIN, Phys. Rev. **111**, 329 (1958), appendix II.

we find that in the center of mass system

$$\begin{aligned} q &= (\boldsymbol{q}, \omega) \qquad q' = (\boldsymbol{q}', \omega) \\ p &= (-\boldsymbol{q}, E) \quad p' = (-\boldsymbol{q}', E) \end{aligned} \tag{A.4}$$

with $\varrho_2 = 0$, $\varrho_2^2 = 1$, $\varrho_3 = 1$, $\varrho_3^2 = 1$ for $(r, s) = (1, 2)$ we get

$$\begin{aligned} \bar{u}(p')\,u(p) &= \frac{1}{2M(M+E)}\,(M + i\varrho_2\boldsymbol{\sigma}\cdot\boldsymbol{q}' + \varrho_3 E)\,(M + i\varrho_2\boldsymbol{\sigma}\cdot\boldsymbol{q} + \varrho_3 E) = \\ &= \frac{M+E}{2M} - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{q}'\,\boldsymbol{\sigma}\cdot\boldsymbol{q}}{2M(M+E)} \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} &\bar{u}(p')\,i\gamma\,\frac{q+q'}{2}\,u(p) = \\ &= \frac{1}{2M(M+E)}(M + i\varrho_2\boldsymbol{\sigma}\cdot\boldsymbol{q}' + \varrho_3 E)\left(i\varrho_2\boldsymbol{\sigma}\cdot\frac{\boldsymbol{q}+\boldsymbol{q}'}{2} - \varrho_3\omega\right)(M + i\varrho_2\boldsymbol{\sigma}\cdot\boldsymbol{q} + \varrho_3 E) = \\ &= -\frac{(E+M)\,(W-M)}{2M} - \frac{M+W}{2M(M+E)}\,\boldsymbol{\sigma}\cdot\boldsymbol{q}'\,\boldsymbol{\sigma}\cdot\boldsymbol{q} \end{aligned} \tag{A.6}$$

so that

$$T = -\frac{E+M}{2M}\,(A + (W-M)B) + \frac{E-M}{2M}\,(A - (W+M)\,B)\,\boldsymbol{\sigma}\cdot\boldsymbol{q}'\,\boldsymbol{\sigma}\cdot\boldsymbol{q} \tag{A.7}$$

The scattering amplitude is

$$F(W, x) = f_1(W, x) + \boldsymbol{\sigma}\cdot\boldsymbol{q}'\,\boldsymbol{\sigma}\cdot\boldsymbol{q}\,f_2(W, x) \tag{A.8}$$

with

$$\begin{aligned} f_1(W, x) &= \frac{E+M}{8\pi W}\,[A(W, x) + (W-M)\,B(W, x)] \\ f_2(W, x) &= -\frac{E-M}{8\pi W}\,[A(W, x) - (W+M)\,B(W, x)] \end{aligned} \tag{A.9}$$

and $x = \cos\theta = \hat{q}\cdot\hat{q}'$.

To carry out the partial wave expansion we remark that the interaction in a particular orbital angular momentum state, l, contributes to scattering in two total angular momentum states, viz. $j = l \pm {}^1/_2$. In general we have different amplitudes in the two cases, so that we write

$$F(W, x) = \sum_{l=0}^{\infty}(2l+1)\,(a_{l-}(W)\,\Pi_{j=l-{}^1/_2} + a_{l+}(W)\,\Pi_{j=l+{}^1/_2})\,P_l(x) \tag{A.10}$$

where the projection operators

$$\Pi_{j=l-{}^1/_2} = \frac{l - \boldsymbol{\sigma}\cdot\boldsymbol{L}}{2l+1}\,;\quad \Pi_{j=l+{}^1/_2} = \frac{l + 1 + \boldsymbol{\sigma}\cdot\boldsymbol{L}}{2l+1} \tag{A.11}$$

are constructed with the aid of the identity

$$j(j+1) = l(l+1) + {}^{3}/_{4} + \boldsymbol{\sigma} \cdot \boldsymbol{L}. \tag{A.12}$$

Thus

$$F(W,x) = \sum_{l=0}^{\infty} (l a_{l-} + (l+1)\, a_{l+})\, P_l(x) + (a_{l+} - a_{l-})\, \boldsymbol{\sigma} \cdot \boldsymbol{L}\, P_l(x)).$$

With the help of the relations

$$\boldsymbol{\sigma} \cdot \boldsymbol{L}\, P_l(x) = (x - \boldsymbol{\sigma} \cdot \hat{q}'\, \boldsymbol{\sigma} \cdot \hat{q})\, P_l'(x)$$

$$l\, P_l(x) = x\, P_l'(x) - P_{l-1}'(x)$$

$$(l+1)\, P_{l+1}(x) = -x\, P_l'(x) + P_{l+1}'(x)$$

we get

$$F(W,x) = \sum_{l=0}^{\infty} (a_{l+} P_{l+1}'(x) - a_{l-} P_{l-1}'(x)) + \boldsymbol{\sigma} \cdot \hat{q}'\, \boldsymbol{\sigma} \cdot \hat{q} \sum_{l=0}^{\infty} (a_{l-} - a_{l+})\, P_l'(x) \tag{A.13}$$

where the general form of the amplitudes $a_{l\pm}$ is

$$a_{l\pm}(W) = \frac{e^{i\delta_{l\pm}(W)} \sin \delta_{l\pm}(W)}{q}. \tag{A.14}$$

To separate out the individual partial waves we write

$$f_1(W,x) = \sum_l a_{l+} (P_{l+1}' - P_{l-1}') - \sum_l (a_{l-} - a_{l+})\, P_{l-1}' \tag{A.15}$$

and use

$$P_{l+1}' - P_{l-1}' = (2l+1)\, P_l; \quad x\, P_l' - P_{l-1}' = l\, P_l \tag{A.16}$$

to get

$$x\, f_2(W,x) + f_1(W,x) = \sum (2l+1)\, a_{l+} P_l(x) + \sum (a_{l-} - a_{l+})\, l\, P_l(x) \tag{A.17}$$

so that

$$\frac{1}{2} \int_{-1}^{1} dx\, P_l(x)\, (x f_2 + f_1) = \frac{l+1}{2l+1}\, a_{l+} + \frac{l}{2l+1}\, a_{l-}. \tag{A.18}$$

Next we make use of

$$\frac{1}{2} \int_{-1}^{1} dx\, (P_{l-1} - P_{l+1})\, P_m' = \delta_{lm} \tag{A.19}$$

to obtain

$$\frac{1}{2} \int_{-1}^{1} dx\, (P_{l-1} - P_{l+1})\, f_2(W,x) = a_{-} - a_{l+}. \tag{A.20}$$

Since

$$x\, P_l = \frac{l}{2l+1}\, P_{l-1} + \frac{l+1}{2l+1}\, P_{l+1} \tag{A.21}$$

we have (A.18)

$$\frac{1}{2}\int\limits_{-1}^{1} dx\, f_1 P_l + \frac{1}{2}\,\frac{l+1}{2l+1}\int\limits_{-1}^{1} dx\, f_2\, P_{l+1} + \frac{1}{2}\,\frac{l}{2l+1}\int\limits_{-1}^{1} dx\, f_2\, P_{l-1} =$$

$$= \frac{l+1}{2l+1}\, a_{l+} + \frac{l}{2l+1}\, a_{l-}. \qquad \text{(A.22)}$$

Also

$$\frac{1}{2}\int\limits_{-1}^{1} dx\, f_2\, P_{l-1} - \frac{1}{2}\int\limits_{-1}^{1} dx\, f_2\, P_{l+1} = a_{l-} - a_{l+}.$$

Therefore

$$\begin{aligned} a_{l-} &= \frac{1}{2}\int\limits_{-1}^{1} dx\, [f_1(W,x)\, P_l(x) + f_2(W,x)\, P_{l-1}(x)] \\ a_{l+} &= \frac{1}{2}\int\limits_{-1}^{1} dx\, [f_1(W,x)\, P_l(x) + f_2(W,x)\, P_{l+1}(x)]. \end{aligned} \qquad \text{(A.23)}$$

For use in sec. IX we record the following formulae

$$\begin{aligned} A(W,x) &= 4\pi\,\frac{W+M}{E+M}\, f_1(W,x) - 4\pi\,\frac{W-M}{E-M}\, f_2(W,x) \\ B(W,x) &= 4\pi\,\frac{1}{E+M}\, f_1(W,x) + 4\pi\,\frac{1}{E-M}\, f_2(W,x). \end{aligned} \qquad \text{(A.24)}$$

Using the relations

$$f^{\pm}_{\substack{1\\2}}(W,x) = \frac{1}{3}\left(f^{(1/2)}_{\substack{1\\2}} + \binom{2}{-1} f^{(3/2)}_{\substack{1\\2}}\right)$$

we can obtain the following expressions

$$\begin{aligned} f_1^{+}(W,x) &\cong \frac{1}{3}\Big(a_1 + 2a_3 - d_{13} - 2d_{33} + d_{15} + 2d_{35} + 3\cos\theta\,(a_{13} + 2a_{33}) + \\ &\qquad + 5\left(\frac{3\cos^2\theta - 1}{2}\right)(d_{15} + 2d_{35})\Big) \\ f_1^{-}(W,x) &\cong \frac{1}{3}\Big(a_1 - a_3 - d_{13} + 2d_{33} + d_{15} - d_{35} + 3\cos\theta\,(a_{13} - a_{33}) + \\ &\qquad + 5\left(\frac{3\cos^2\theta - 1}{2}\right)(d_{15} - d_{35})\Big) \\ f_2^{+}(W,x) &\cong \frac{1}{3}\,(a_{11} + 2a_{31} - a_{13} - 2a_{33} + 3\cos\theta\,(d_{13} + 2d_{33} - d_{15} - 2d_{35})) \\ f_2^{-}(W,x) &\cong \frac{1}{3}\,(a_{11} - a_{31} - a_{13} + a_{33} + 3\cos\theta\,(d_{13} - d_{33} - d_{15} + d_{35})) \end{aligned} \qquad \text{(A.25)}$$

where a_{2i} are the amplitudes in the state of i-spin "i", angular momentum $^1/_2$ and orbital angular momentum 0, $a_{2i,2j}$ are the amplitudes of i-spin "i" angular momentum "j" and orbital angular momentum 1, and $d_{2i,2j}$ are the same amplitudes with orbital angular momentum 2. The usual notation is

$$\begin{aligned} a_{2i} &= \frac{e^{i\alpha_{2i}} \sin \alpha_{2i}}{q} \\ a_{2i,2j} &= \frac{e^{i\alpha_{2i,2j}} \sin \alpha_{2i,2j}}{q} \\ d_{2i,2j} &= \frac{e^{i\delta_{2i,2j}} \sin \delta_{2i,2j}}{q}. \end{aligned} \tag{A.26}$$

Appendix B

Isotopic Spin Decomposition

a) Pion-Nucleon Scattering

We are given

$$F_{\beta\alpha} = \delta_{\beta\alpha} F^{(+)} + \frac{1}{2} [\tau_\beta, \tau_\alpha] F^{(-)} \tag{B.1}$$

and we wish to express $F^{(\pm)}$ in terms of $F^{(i)} (i = {}^1/_2, {}^3/_2)$.
We write

$$F_{\beta\alpha} = \sum_{i={}^1/_2}^{{}^3/_2} F^{(i)} (\Pi^{(i)})_{\beta\alpha} \tag{B.2}$$

where the projection operators are

$$\Pi^{({}^1/_2)} = \frac{1}{3} (1 - \tau \cdot I_\pi); \quad \Pi^{({}^3/_2)} = \frac{1}{3} (2 + \tau \cdot I_\pi). \tag{B.3}$$

Thus

$$F_{\beta\alpha} = \frac{1}{3} (F^{({}^1/_2)} + 2F^{({}^3/_2)}) \delta_{\beta\alpha} + \frac{1}{3} (F^{({}^3/_2)} - F^{({}^1/_2)}) (\tau \cdot I_\pi)_{\beta\alpha}. \tag{B.4}$$

Now using

$$(I_{\pi\varkappa})_{\beta\alpha} = - i \varepsilon_{\varkappa\beta\alpha} \tag{B.5}$$

we obtain

$$(\tau \cdot I_\pi)_{\beta\alpha} = - i \varepsilon_{\varkappa\beta\alpha} \tau_\varkappa = - \frac{1}{2} [\tau_\beta, \tau_\alpha]. \tag{B.6}$$

Hence

$$F_{\beta\alpha} = \frac{1}{3} (F^{({}^1/_2)} + 2F^{({}^3/_2)}) \delta_{\beta\alpha} + \frac{1}{2} [\tau_\beta, \tau_\alpha] \frac{1}{3} (F^{({}^1/_2)} - F^{({}^3/_2)}) \tag{B.7}$$

so that

$$\begin{aligned} F^{(+)} &= \frac{1}{3} (F^{({}^1/_2)} + 2F^{({}^3/_2)}) \\ F^{(-)} &= \frac{1}{3} (F^{({}^1/_2)} - F^{({}^3/_2)}). \end{aligned} \tag{B.8}$$

b) Pion-Pion-Scattering

We again write

$$A_{\gamma\alpha,\,\delta\beta} = \sum_{i=0}^{2} A^{(i)} (P^{(i)})_{\gamma\alpha,\,\delta\beta} \tag{B.9}$$

with the projection operators

$$\begin{aligned} P^{(0)} &= \frac{1}{3} (\boldsymbol{I}_1 \cdot \boldsymbol{I}_2 + 1)(\boldsymbol{I}_1 \cdot \boldsymbol{I}_2 - 1) \\ P^{(1)} &= \frac{1}{2} (\boldsymbol{I}_1 \cdot \boldsymbol{I}_2 + 2)(1 - \boldsymbol{I}_1 \cdot \boldsymbol{I}_2) \\ P^{(2)} &= \frac{1}{6} (\boldsymbol{I}_1 \cdot \boldsymbol{I}_2 + 1)(\boldsymbol{I}_1 \cdot \boldsymbol{I}_2 + 2). \end{aligned} \tag{B.10}$$

Again using $(I_\varkappa)_{\beta\alpha} = -i\varepsilon_{\varkappa\beta\alpha}$ we get

$$\begin{aligned} (I_1 \cdot I_2)_{\gamma\alpha,\,\delta\beta} &= (-i\varepsilon_{\varkappa\gamma\alpha})(-i\varepsilon_{\varkappa\delta\beta}) \\ &= \delta_{\gamma\beta}\delta_{\alpha\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta} \end{aligned} \tag{B.11}$$

and

$$\begin{aligned} (I_1 \cdot I_2)^2_{\gamma\alpha,\,\delta\beta} &= (I^{(1)}_\varkappa I^{(1)}_l)_{\gamma\alpha} (I^{(2)}_\varkappa I^{(2)}_l)_{\delta\beta} \\ &= (I^{(1)}_\varkappa)_{\gamma\varrho} (I^{(1)}_l)_{\varrho\alpha} (I^{(2)}_\varkappa)_{\delta\sigma} (I^{(2)}_l)_{\sigma\beta} \\ &= \varepsilon_{\varkappa\gamma\varrho}\, \varepsilon_{l\varrho\alpha}\, \varepsilon_{\varkappa\delta\sigma}\, \varepsilon_{l\sigma\beta} \\ &= \delta_{\gamma\delta}\, \delta_{\alpha\beta} + \delta_{\alpha\gamma}\, \delta_{\beta\delta} \end{aligned}$$

so that

$$\begin{aligned} A_{\gamma\alpha,\delta\beta} = \frac{1}{3}\, \delta_{\alpha\beta}\, \delta_{\gamma\delta}\, (A^{(0)} - A^{(2)}) + \frac{1}{2}\, \delta_{\alpha\gamma}\, \delta_{\beta\delta}\, (A^{(1)} + A^{(2)}) - \\ - \frac{1}{2}\, \delta_{\alpha\delta}\, \delta_{\beta\gamma}\, (A^{(1)} - A^{(2)}). \end{aligned} \tag{B.13}$$

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